Portfolio Choice with Illiquid Assets

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Harvard Endowment: A Cautionary Example

Harvard Endowment

- Performance of Harvard endowment June 2008 to June 2009: -27.3%. Fund shrank from $36.9 billion to $26.0 billion [Note S&P500 performance was -30% during this period]

- At June 2008, endowment distributions totaled $1.2 billion, representing 34% of the University’s $3.5 billion revenue. For some schools, the reliance on the endowment was even higher:
  
  Radcliffe 83%
  Faculty of Arts and Sciences 52%
  Law 37%
  Business 20%

- Spending rate (payout rule) is variable, but it is smooth and at June 2008 was 4.8%
Harvard Endowment

- Harvard was an early adopter of the “endowment” model based on diversification concepts extended to illiquid assets (thanks to Swensen, Leibowitz, and others)

<table>
<thead>
<tr>
<th>Harvard Endowment Asset Allocation Pre-Financial Crisis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Policy Portfolio</td>
</tr>
<tr>
<td>------------------</td>
</tr>
<tr>
<td>Liquid</td>
</tr>
<tr>
<td>Semi-Liquid</td>
</tr>
<tr>
<td>Illiquid</td>
</tr>
<tr>
<td>Total</td>
</tr>
</tbody>
</table>

- The losses from the financial crisis mean Harvard’s budget has to shrink by approximately 20%, not including the massive cash outflows Harvard is taking from its swap positions. Harvard found out it can’t “eat” illiquid assets!
Portfolio Choice with Illiquid Assets
● Standard asset allocation models assume that investors have the ability to freely rebalance their portfolios at any time (sometimes at a cost)

● However, some assets cannot be traded, at any price, for significant lengths of time:
  – Direct real estate investments
  – Private equity and venture capital
  – Certain securitized fixed income and structured credit products

● How does illiquidity affect asset allocation?
Model

Assets:

- Riskless bond, interest rate $r$, freely tradeable
- Liquid risky asset [equity], freely tradeable
- Illiquid risky asset. Tradeable only at random times $\tau \sim \text{Poisson} (\lambda)$. The expected waiting time between rebalancing is $1/\lambda$. More illiquid assets have lower $\lambda$.

Notation: $W =$ total wealth, $X =$ illiquid asset wealth
Model

Preferences:
● CRRA utility over consumption

Outputs:
● Optimal asset holdings: liquid and illiquid asset holdings, risk-free holdings
● Optimal consumption
• Value function  \( F(W_t, X_t) = \max E \left[ \int_0^\infty e^{-\beta t} \frac{C_t^{1-\gamma}}{1-\gamma} \right] \Rightarrow W_t^{1-\gamma} G \left( \frac{X_t}{X_t + W_t} \right) \)

• When a trading time arrives, there is a jump in the agent’s continuation value as he rebalances \( F \rightarrow F^* \)

\[
F^*(W_t, X_t) = \max_{I \in [-X_t, W_t]} F(W_t - I, X_t + I) = K_\lambda (W_t + X_t)^{1-\gamma}
\]

• Three unknowns: \( G(\cdot), K_\lambda, \left( \frac{X_t}{X_t + W_t} \right)^* \)

• Three conditions:
  
  – Hamilton-Jacobi-Bellman
  
  – Value matching. At \( \left( \frac{X_t}{X_t + W_t} \right)^* \), \( F = F^* \)
  
  – Smooth pasting. At \( \left( \frac{X_t}{X_t + W_t} \right)^* \), \( F_W = F_W^* \)
Illiquidity-Induced Endogenous Risk Aversion

- The presence of illiquidity induces time-varying, endogenous risk aversion

- Intuition:

  In a standard Merton problem where both assets are always tradeable, an agent only cares about total wealth. The risk is that total wealth goes to zero and the agent cannot consume.

  The agent can only consume out of liquid wealth. Therefore, with illiquid and liquid assets he also cares about the risk of liquid wealth going to zero.

- The ratio of liquid to total wealth becomes a state variable. That is, effective risk aversion depends on liquidity solvency ratios.
Effective Relative Risk Aversion (RRA)

Optimal rebalancing point

RRA wrt total wealth

RRA wrt liquid wealth

RRA wrt illiquid wealth

$\frac{X}{(W+X)}^*$

$\frac{X}{(X+W)}$ Proportion of Illiquid Assets
Illiquid Asset Holdings

- Illiquidity markedly reduces optimal holdings relative to the Merton benchmark. Furthermore, illiquid asset holdings are very skewed.

<table>
<thead>
<tr>
<th>Average Turnover</th>
<th>$\lambda$</th>
<th>Rebalance Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 years</td>
<td>0.1</td>
<td>0.05</td>
</tr>
<tr>
<td>5 years</td>
<td>0.2</td>
<td>0.11</td>
</tr>
<tr>
<td>2 years</td>
<td>0.5</td>
<td>0.24</td>
</tr>
<tr>
<td>1 year</td>
<td>1.0</td>
<td>0.37</td>
</tr>
<tr>
<td>$\frac{1}{2}$ year</td>
<td>2.0</td>
<td>0.44</td>
</tr>
<tr>
<td>Continuously</td>
<td>$\infty$</td>
<td>0.59</td>
</tr>
</tbody>
</table>
Distribution of Illiquid Holdings

Optimal rebalancing point

\[ \frac{x^*}{1 + x^*} \]

Proportion of Illiquid Assets
Optimal Consumption

Optimal rebalancing point

Merton consumption: 2 assets

Merton consumption: 1 asset

Fraction of liquid wealth

Fraction of total wealth

Proportion of Illiquid Assets
Correlation

- In the presence of illiquidity, “near-arbitrage” opportunities arising from high correlations are not exploited. There is no “arbitrage” because illiquid and liquid assets are not close substitutes.

- Note: Merton positions at $\rho = 1$ are +/- infinity.
How much does an investor need to be compensated for illiquidity? To be able to trade the illiquid asset continuously, an investor requires liquidity premiums of:

<table>
<thead>
<tr>
<th>Average Turnover</th>
<th>( \lambda )</th>
<th>Illiquidity Premium</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 years</td>
<td>0.1</td>
<td>0.060</td>
</tr>
<tr>
<td>5 years</td>
<td>0.2</td>
<td>0.043</td>
</tr>
<tr>
<td>2 years</td>
<td>0.5</td>
<td>0.020</td>
</tr>
<tr>
<td>1 year</td>
<td>1.0</td>
<td>0.009</td>
</tr>
<tr>
<td>½ year</td>
<td>2.0</td>
<td>0.007</td>
</tr>
</tbody>
</table>
Conclusion

● The periodic inability to trade (illiquidity) has a large impact, for realistic parameters, on portfolio choice
  – Lower holdings of both the liquid and illiquid assets relative to the Merton benchmark of continuous trading for all assets
  – Consumption is lower than Merton

● Illiquidity risk induces time-varying risk aversion which is greater than the constant risk aversion coefficient of utility. This is because illiquid assets cannot be used to fund immediate consumption.

● Extensions: Time-varying illiquidity risk (regimes); Correlated frequency of trading with asset returns; Investors pay a cost to increase liquidity (optimal transactions costs).