Andrew Ang, Managing Director, BlackRock Inc., New York, NY
Andrew.Ang@BlackRock.com

Ked Hogan, Managing Director, BlackRock Inc., New York, NY
Ked.Hogan@BlackRock.com

Sara Shores, Managing Director, BlackRock Inc, New York, NY
Sara.Shores@BlackRock.com

Factor Risk Premiums and Invested Capital: Calculations with Stochastic Discount Factors

Andrew Ang, Ked Hogan, and Sara Shores

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Portfolios with positive exposures to rewarded risk premiums have historically exhibited high average returns adjusting for their market betas. As capital allocated to such strategies increases, the excess returns of these portfolios should decrease. We compute the flows from low-return to high-return portfolios required so that the factor risk premiums are equal to zero. We also estimate the factor premiums resulting when all capital from the bottom 30% of stocks ranked by common risk factors—value, size, momentum, and idiosyncratic volatility—is transferred to the top 30% of stocks. We find that size is the least robust factor and in fact reverses under this scenario. The value, momentum, and volatility factor premiums are reduced by at most half from their historical premiums.

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There are several well-known characteristics or factor exposures which are related to the cross section of expected returns.[1] For example, value stocks with high prices relative to intrinsic value have tended to have higher risk-adjusted returns than growth stocks. If investors were to sell the lower-return growth stocks and purchase higher-return value stocks, the relative price of the growth portfolio would likely fall and the risk-adjusted expected return of growth stocks would likely increase. At the same time, the movement of capital from growth to value portfolios would tend to push up the relative prices of value stocks, causing the risk-adjusted expected returns of value stocks to decrease.

In this article, we estimate the required movement of capital of stocks with negative factor exposures, like growth stocks which have had low risk-adjusted returns, to portfolios with high exposures to rewarded factors, like value stocks which have had high risk-adjusted returns, so that after this transfer of capital there are no longer any anomalous factor risk premiums. We show the relation between the capital flow and excess expected returns of various factor strategies applying the stochastic discount factor, or pricing kernel, methodology of Hansen and Jagannathan (1991).[2] The optimal pricing kernel portfolio perfectly prices all assets, which is represented by a perfectly efficient portfolio. There are no factor risk premiums or "alphas" with respect to that efficient portfolio.

We interpret the difference between the optimal pricing kernel and the capitalization weight market portfolio (often abbreviated to just the "market") as a completion portfolio. That is, the efficient portfolio is equal to the market portfolio plus a completion portfolio; the completion portfolio transfers, after adjusting for risk, capital from low-return portfolios to high-return strategies. This is intuitive because in the CAPM equilibrium, value stocks have "too high" a risk-adjusted return and growth stocks have "too low" a return. To move to the equilibrium with no factor risk premiums represented by the stochastic discount factor, capital must be transferred to the high-return value portfolio, lowering its risk-adjusted return. Investors purchase value stocks by divesting from low-return growth stocks. Importantly, the completion portfolio is zero cost—the portfolio weights sum to zero—so, an interpretation of the completion portfolio is that it represents the set of flows from low-return to high-return strategies that must take place in order for the factor risk premiums to disappear.

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[1] Ang (2014) contains a comprehensive summary of common factor risk premiums with their economic rationales.

[2] Following the literature, we use the words "stochastic discount factor" and "pricing kernel" interchangeably.

For the factors that we examine—value, size, momentum, profitability, and idiosyncratic volatility—the pricing kernel is a levered portfolio. Put another way, the completion portfolio sells a larger amount of growth stocks than the current market capitalization of growth stocks. This illustrates how large the factor risk premiums are relative to the pricing model of the CAPM.

We compute the reduction in the expected value premium for any movement of capital from growth stocks to value stocks. (Note the optimal pricing kernel where there is no value premium is a special case of a general capital transfer.) For each factor, we examine portfolios where all capital is moved from the bottom 30% of stocks with the lowest factor returns (growth, say) to the top 30% of stocks with the highest factor returns (value). The reduction is largest for the value premium, from 4.14% to 2.25% per year. In the same exercise for idiosyncratic volatility portfolios, we find that the low volatility risk premium drops slightly from 5.16% to 4.55% per year. There is a similar insignificant effect on momentum and quality: applying the same transfer of capital from the low-return to the high-return factor portfolios, the momentum premium reduces from 7.93% to 6.42% and the profitability premium reduces from 3.18% to 2.91%. However, the size premium is the least robust and could turn negative after sizable capital moves from large to small stocks.

We interpret the Hansen-Jagannathan (1991) completion portfolio as a measure of capacity because it represents the capital transfer required to attenuate the factor risk premiums. Other measures of capacity consider average exposures of various investor clienteles, and capacity can be represented as exposure to a given factor being taken up by one clientele while another type of clientele disinvests from that strategy. This approach, such as taken by Madhavan, Sobczyk, and Ang (2016) interprets the magnitude of a factor loading, but implicitly holds the factor premium fixed. Transaction cost estimates of capacity done by Ratcliffe, Miranda, and Ang (2016) and Novy-Marx and Velikov (2016) are also partial equilibrium approaches, and implicitly assume that both the factor loadings and premiums are constant. While we believe that transaction costs are extremely important in practice, we do not consider them in this paper. Our focus is on the theoretical link between the expected factor premiums and the capital allocated to them which is made endogenous through the stochastic discount factor.

Data

We examine portfolios sorted on book-to-market ratios (representing the value-growth premium), market capitalization (representing the size premium), past 12 month returns (representing the momentum premium), quality as measured by profitability (which we refer to as the profitability premium), and idiosyncratic volatility relative to the Fama and French (1993) model (representing the low volatility premium).[3] In all cases, we work with three hypothetical portfolios representing the low-return, neutral, and high-return portfolios. The breakpoints are at the 30th percentile and 70th percentile for the sorting variable. All portfolios are market capitalization weighted. Three portfolios are the minimum number required to compute our results (recall there are *N-1* degrees of freedom in *N* portfolio weights). Representing the factor premiums using three portfolios also makes our results highly conservative because using quintile or decile portfolios significantly increases the spread in the portfolio returns, and hence factor alphas.

Our focus is on relative pricing. We take the test portfolio returns as given and compute alphas of test assets relative to the CAPM and other stochastic discount factors. The factor premiums are all then expressed in relative terms: the differences the alpha of the value portfolio minus the alpha of the growth portfolio. In the efficient portfolio with the optimal stochastic discount factor, the difference in alphas is equal to zero.

Methodology

We work in gross returns and take each set of factor portfolios separately as test assets. We denote the vector of portfolio gross returns by R with market capitalization weights w_{mkt}. By construction, w_{mkt} sums to 1. The market-capitalization weight (or cap weight) market portfolio, R_{mkt}, is given by

$$R_{mkt} = w_{mkt} \cdot R.$$

CAPM

The Sharpe (1964) beta-pricing relation is given by:

$$E(R) - R_f = \beta \cdot \lambda_{mkt} = \frac{cov(R, R_{mkt})}{var(R_{mkt})} \cdot \left[E(R_{mkt}) - R_f\right], \tag{1}$$

where R_f is the gross risk-free rate, $\beta = cov(R, R_{mkt})/var(R_{mkt})$ is the beta or exposure to the market portfolio, and $\lambda_{mkt} = E(R_{mkt}) - R_f$ is the risk premium of the market factor.

An enormous literature[4] shows that the market portfolio is inefficient, or that there are alpha (α) pricing discrepancies using the market portfolio as the pricing factor. That is, empirically we

[3] Unless otherwise stated, the data underlying the analysis in this paper is the Fama French factors, constructed by Kenneth French using the time periods of July 1963-December 2015 and is available at http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html
[4] Ang (2014)

find that

$$\bar{R} - R_f = a + \beta \cdot [\bar{R}_{mkt} - R_f], \tag{2}$$

where $\bar{R}$ and $\bar{R}_{mkt}$ are the average historical returns of the test portfolios and the market portfolio, respectively. Alpha is positive for the rewarded factor portfolio, such as value, and negative for the portfolio with the opposite factor exposure, which is growth. The portfolio alphas are associated with observed market weights in the value, growth, and neutral portfolios, which are given by w_{mkt}.

Stochastic Discount Factors

Following Hansen and Jagannathan (1991), the CAPM is a special case of a stochastic discount factor or pricing kernel *m*, where we can generalize equation (1) to express the excess return of an asset as its exposure, γ, to the pricing kernel multiplied by the price of risk of the pricing kernel, λ (see Cochrane, 2001):[5]

$$E(R) - R_f = \gamma \cdot \lambda = \frac{cov(R,m)}{var(m)} \cdot \left[-\frac{var(m)}{E(m)}\right] \cdot \tag{3}$$

The CAPM is a special pricing kernel, and equation (3) simplifies to the CAPM by setting $m = m_{mkt}$, where

$$m_{mkt} = a + bR_{mkt}, \tag{4}$$

with the constants *a* and *b* given by

$$a = \frac{1}{R_f} - b\mu_m$$

$$b = -\frac{\mu_{mkt} - R_f}{R_f \sigma^2{}_{mkt}} \cdot$$

The CAPM pricing kernel in equation (4) corresponds to the set of market cap weights, w_{mkt}, so that $\mu_{mkt} = w_{mkt} \cdot E(R)$ and $\sigma^w_{mkt} = var(w_{mkt} \cdot R)$.

There is an optimal pricing kernel for which all portfolio alphas are equal to zero, or there are no excess factor premiums. We denote this optimal kernel by *m**. Hansen and Jaganna-

[5] An economic interpretation of the pricing kernel, *m*, is that it represents an index of "bad times" (see Ang, 2014), which accounts for the negative sign in the price of risk of the pricing kernel, λ. Equation (3) says that if an asset's covariance with respect to the pricing kernel is high, then its expected return is low; assets with payoffs that tend to be high during bad times are valuable and a representative investor does not require high expected returns to hold those assets because of their hedging properties. The mean of the pricing kernel determines the risk-free rate, with $E(m) = 1/R_f$.

than (1991) show that this stochastic discount factor can be written as a function of the portfolio test assets:

$$m^* = w^* \cdot R, \tag{5}$$

and the weights w^* correspond to an efficient, or tangency portfolio, on an mean-variance frontier. In the Appendix, we detail the calculation of the optimal pricing kernel weights in equation (5). In an "optimal market" portfolio with weights w^*, there are no alphas for any asset and equation (3) holds exactly.

Completion Portfolios

The optimal pricing kernel weights can be written as the market-cap weights plus a completion portfolio:

$$w^* = w_{mkt} + (w^* - w_{mkt}), \tag{6}$$

where we start from market weights and add the completion portfolio, $(w^* - w_{mkt})$, to obtain the efficient pricing kernel.

The completion portfolio is zero cost. That is, $(w^* - w_{mkt})$ sums to zero. With market capitalization portfolio weights alone, w^*, there are positive (negative) alphas for value (growth) stocks. The completion portfolio corrects for this by moving capital from portfolios with low risk-adjusted excess returns (which have prices that are too high) to portfolios with high risk-adjusted excess returns (which have corresponding prices that are too low). The amount of the capital is also determined by a covariance risk adjustment. The capital transfer in the completion portfolio is the minimum necessary so that the alphas are equal to zero in equation (3). Below, we show that the factor premiums are so large in historical data that the completion portfolio is highly levered and contains weights greater in absolute value than the market-cap weights.

We also consider completion portfolios which do not attain perfect efficiency, but involve a partial transfer from portfolios with negative premiums, like growth, to positive premiums, like value. In particular, we consider a completion portfolio which transfers all capital from growth to value. This partial completion portfolio reduces the factor alphas, but does not eliminate them. Although we use a strict value of zero market cap in the growth portfolio for computation, this theoretical exercise can be viewed as an epsilon, or *de minimus*, holding of growth stocks. As the growth AUM decreases, the risk-adjusted expected return of growth stocks increases. The pricing kernel corresponding to these partial completion portfolios is given by equation (4) for appropriately redefined weights.

The completion portfolio can be interpreted to be a stochastic discount factor estimate of factor capacity. The pricing kernel formulation simultaneously takes into account the expected return (or price) and the amount of capital (quantity) allocated to each portfolio. All else equal, the larger the weight or the greater the supply of capital, the higher the relative price of that portfolio. Equivalently, the higher relative price translates to a lower risk-adjusted expected return.

Results for Value-Growth Portfolios

We start by illustrating the methodology and report the market weight, completion, and tangency weights for value portfolios.

Value-Growth Weights

Exhibit 1 graphs the market cap weights of the growth and value portfolios from July 1963 to December 2015. The remainder represents the weight in the neutral portfolio. The portfolio weights to growth and value are relatively stable, and the average growth, neutral, and value weights are 0.519, 0.336, and 0.146, respectively. We take these weights as the market portfolio. Note that as is well known, value stocks tend to have a small size bias (see Loughran, 1997). Over this sample, the risk-free rate is 4.8%, which we fix. We examine lower risk free rates, which are observed in more recent data, in robustness tests below.

Value-Growth Efficient Frontier

In Exhibit 2, we plot the efficient frontier for the value-growth portfolios. The market portfolio illustrated by the red triangle lies in the interior of the mean-variance frontier; its inefficiency gives rise to a positive alpha for value premium:

	Portfolio Alphas				
Pricing Kernel	Growth	Neutral	Value		Value - Growth
Cap Weight Market	-1.18%	0.55%	2.96%		4.14%

Exhibit 2 plots the tangency portfolio in the circle, which is the risky portfolio with the highest empirical Sharpe ratio of 0.622. In comparison, the Sharpe ratio of the capital weighted portfolio is 0.446. The line connecting the tangency portfolio with the risk-free rate intercept is the

Capital Allocation Line (CAL), in the language of the CAPM.[6] When we use the tangency portfolio as the pricing kernel, all portfolio alphas are equal to zero.

In Exhibit 3, we zoom in on the mean-variance frontier by changing the axes' scales. We start with the market portfolio, which produces a value premium of 4.14% and has weights:

	Market Weights
Growth	0.519
Neutral	0.336
Value	0.146

The alphas from the tangency portfolio are equal to zero, but this efficient portfolio is a highly levered portfolio:

	Efficient Weights
Growth	-0.334
Neutral	-0.746
Value	2.080

The efficient portfolio has a very large position of 208% in value because of its high return, and shorts growth and neutral stocks. Note that the efficient portfolio is also more volatile than the market, with a standard deviation of 20.4% compared to the market's volatility of 15.1%. Economically, the alphas can only be set to zero if the average, or representative, investor takes additional risk. The completion portfolio specifies exactly what additional risk is needed to drive the alphas to zero.[7]

Value-Growth Completion Portfolio

We can move from the market portfolio to the efficient portfolio (optimal pricing kernel) via the completion portfolio, which is a zero-cost portfolio and transfers from low-return assets (growth and neutral) to high-return assets (value):

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[6] Theoretically, an investor should use leverage to seek expected returns higher than the tangency portfolio by taking positions in the tangency portfolio and shorting risk-free assets, and by doing so lying on the CAL. This is one of the central concepts involved in "risk parity" styles of investing.

[7] The distance between the market portfolio and the efficient portfolio is a version of the Hansen and Jagannathan (1997) distance metric, which can be interpreted as a norm of the completion portfolio.

	Weights		
	Market	Efficient	Completion
Growth	0.519	-0.334	-0.853
Neutral	0.336	-0.746	-1.082
Value	0.146	2.080	1.934
Sum	1.000	1.000	0.000

The completion portfolio is highly levered because the value premium is relatively large in the historical data. Put another way, the value premium represents an under-ownership of value stocks—and the amount of transfers to value strategies need to increase the holdings of value stocks by 1300% (the portfolio weight must increase from 0.146 to 2.080) to remove the value premium! The completion portfolio is so levered because it needs to add both additional returns and volatility to the market portfolio to obtain the efficient portfolio:

	Mean	Stdev
Market	11.53%	15.10%
Efficient	17.46%	20.36%
Completion	5.92%	14.19%

Economic Interpretation

Why does this flow not happen in practice? That is, why does the average investor not allocate more capital to value?

First, the value premium can exist because value stocks bear extra risk and investors overweight growth stocks because they do not want to bear that risk in equilibrium. One such risk is that value firms have more inflexible capital structures (see Berk, Green, and Naik, 1999). The In this explanation, the completion portfolio captures the risk represented by value stocks that needs to be added to the market to perfectly price those assets.

Second, there could be market or behavioral structural impediments that impede the flow of capital from growth to value stocks. These could include investors preferring to hold growth stocks because they irrationally believe that their recent growth will continue at rapid rates (see Lakonishok, Shleifer, and Vishny, 1994). The size of the completion portfolio, especially its levered positions, also implies that the value premium is unlikely to be removed even by substantial shifts from growth to value strategies according to the model.[8]

It is interesting to compare the completion portfolio with Ross (1976) multi-factor models. In particular, the multi-factor model of Fama and French (1993) starts with the market portfolio and then adds additional factors of size and value. The size and value factors are zero cost: the size factor, SMB, stands for small stocks minus big and the HML value factor is for high book-to-market stocks minus low book-to-market stocks. These are special forms of completion portfolio factors which are added to the market factor to account for size and value factor premiums.

Finally, the additional risk represented by the completion portfolio can be interpreted as the extra "hedging demands" of Merton (1971) which help determine the risk premiums of the whole economy in addition to the cap-weight market portfolio factor. This is a manifestation of other sources of risk—some of which can come from the risk characteristics of the rewarded factors themselves, structural impediments, or investors' behavioral biases.

Partial Completion Portfolios

We now consider partial completion portfolios, which still transfer AUM from growth to value stocks but not with the leverage embedded in the optimal pricing kernel. In Exhibit 3, we plot the position of different pricing kernels moving capital from growth to value in the purple triangles. The top purple triangle labeled "Growth wt = 0.00" corresponds to a pricing kernel with a weight of zero in growth, 0.336 in neutral (the same as the market), and 0.664 in value (equal to the original value weight in the market plus the weight of growth). As the transfers from growth to value become larger we approach, but do not reach, the efficient tangency portfolio, and gradually reduce the value premium. When we transfer all of the growth capital to value stocks, the expected value premium is now reduced from 4.14% to 2.25%, which is a reduction in percentage terms of 46%:

	Alphas	
	CAPM	Growth to Value
Growth	-1.18%	-1.71%
Neutral	0.55%	-1.07%
Value	2.96%	0.54%
Value Premium	4.14%	2.25%

[8] A corollary of these findings is that estimates of long-term factor risk premiums using Black and Litterman (1991) methods based only on current market cap weights will produce much lower estimates of factor risk premiums compared to historical data.

Intuitively, the partial transfer from growth to value operates in the same direction as the optimal completion portfolio: reducing the weight of expensive growth stocks with low returns, and increasing the weight of cheap value stocks with high returns. This decreases the value premium. We note that the dollar amount corresponding to this transfer is enormous: the market at December 2015 is $22.5 trillion in this sample and this exercise corresponds to hypothetically transferring $10.5 trillion from growth to value stocks. Clearly, the fact that the value premium still exists implies that capacity for the value risk premium is very large given historical experience.

Robustness to Low Risk-Free Rates

In Exhibit 4, we consider an extreme case of a risk-free rate of zero. Decreasing the risk-free rate increases the Sharpe ratio of the tangency portfolio: in Exhibit 5, the slope of the CAL (or efficient frontier) is higher for the zero risk-free rate and has a Sharpe ratio of 0.89, compared to the Sharpe ratio of 0.62 for the risk-free rate equal to 4.8%. Not surprisingly, the value risk premium is then *larger* with lower risk-free rates because the market portfolio is more inefficient. (The value premium increases to 4.47% for $R_f = 0.0\%$ versus a premium of 4.14% for $R_f = 4.8\%$.) That is, lower risk-free rates require even larger transfers from growth to value to attenuate the value premium.

Results for Momentum, Size, and Idiosyncratic Volatility Portfolios

We summarize the results for all factor portfolios in Exhibit 5. For completeness, the first panel repeats the results for value-growth portfolios discussed in the previous section.

Value

With the additional detail in Exhibit 5, we highlight some more details of the way the stochastic discount factor affects the computation of the factor premium. The overage returns of the test portfolio are given in the first column. The market-capitalization CAPM pricing kernel produces a negative growth alpha of -1.18% and a positive value alpha of 2.96%, producing a value premium of 4.14%. Note that the market is a weighted average of the test portfolios.

The factor premium focuses on the relative pricing of the extreme portfolios. In the partial transfer pricing kernel, where we move all the AUM in growth stocks to value stocks, the value premium shrinks to 2.25%. Note that the actual growth alpha has become more negative, however, from -1.18% under the CAPM to -1.71% in the new stochastic discount factor even as the value premium shrinks. In fact, all the alphas have moved downwards but the difference between the value and growth alphas have decreased. The factor premium is a statement about *relative* pricing.

The intuition behind these results is as follows. The efficient portfolio has a weight in the neutral portfolio to match the *average* return equal to zero. (The efficient portfolio also sums to one and is still a weighted average of the test portfolios.) We have held the neutral AUM constant in the partial transfer pricing kernel, which in this case pushes all of the alphas downwards. In the final column reporting the optimal stochastic discount factor, all three portfolio weights change allowing us to match both *relative* alpha differences and the *average* alpha. The weights are set so that all alphas are equal to zero.

Size

Exhibit 5 shows that over our 1963 to 2015 sample period, there is a size premium of 2.08%, which is about half the size of the value premium (4.14%). (Note that the size premium is larger in raw returns which do not adjust for market risk.[9]) By construction, the market consists of mostly large stocks with a weight of 80.9%.

In the partial completion portfolio, we transfer all the 80.9% weight from large stocks to small. This turns the size premium negative, but there are still large alphas—now positive ones of 0.63% and 0.88% for large and neutral portfolios, respectively. The optimal pricing kernel is still levered and resembles a butterfly portfolio: taking short positions in large and small stocks, and overweighting neutral stocks.

Note that in a transaction cost estimate of capacity, such as done by Ratcliffe, Miranda, and Ang (2016), size is one of the factors with the highest capacity—because it is easiest to trade in terms of low turnover and has a large number of holdings. The stochastic discount factor perspective produces the opposite result because of the small difference in risk-adjusted returns or alphas, between small and large stock portfolios. The large stock portfolio also comprises, by construction, a large capitalization weight and so there is a relatively big effect on relative prices when this amount of capital is transferred in the partial completion portfolio.

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[9] The size premium is smaller when measured using decile portfolios and is statistically insignificant in a formal Gibbons, Ross, and Shanken (1989) test. Small stocks have higher returns, however, as indicated by the first column in Exhibit 5, but much smaller risk-adjusted returns after controlling for market beta.

Momentum

The momentum premium of 7.93% is approximately twice as large the value premium (4.14%) and four times as large as the size premium (2.08%). There is a much less pronounced small-size bias compared to the value factor; loser stocks have an average weight of 21.4% compared with 33.9% for winners.

After moving the capital in loser stocks to winners, the momentum premium shrinks by less than one-fifth, moving from 7.93% to 6.42%. The alphas of winner stocks decrease in the partial completion portfolio, but so do the alphas of losers, and these two effects off-set each other producing the relatively small change in the momentum premium. Again, the stochastic discount factor is levered; as expected, the optimal pricing kernel wants to short losers and long winners, with optimal holdings of -0.682 and 1.620 on losers and winners, respectively.

Profitability

The quality premium, measured by ranking stocks on profitability, is 3.18%, which is approximately one percent lower than the value premium of 4.14% in the tercile portfolio sorts. Most firms are neutral or high profitability firms. After transferring the 16.7% capitalization weight from low profitability to high profitability, the quality premium reduces slightly from 3.18% to 2.91%. The small change in the quality premium is due to the fact that there is a relatively small amount of capital in the low profitability category to begin with.

Idiosyncratic Volatility

The final panel presents the results of idiosyncratic volatility portfolio sorts. Confirming Ang et al. (2006), volatile stocks have very low risk-adjusted returns of -4.28%, whereas neutral and stable stocks have returns more in line with their predicted ones from the CAPM (alphas of 0.55% and 0.88% for neutral and stable stocks, respectively). Volatile stocks tend to be smaller stocks.

In the partial completion portfolio, there is still a large low volatility factor premium: the idiosyncratic volatility premium decreases to 4.55% compared to 5.16% with the regular market portfolio. Not surprisingly, the optimal portfolio holds a very large short position in low-return high volatility stocks, of -1.22, and transfers most of this capital to neutral stocks, with a weight of 2.31.

Conclusion

As capital moves to assets with high risk-adjusted returns, with corresponding low relative prices, from assets with low risk-adjusted returns and high relative prices, the expected risk-adjusted returns of the original high-return portfolio will decrease. In equilibrium, flows and expected return should be inversely related. We apply this concept to measure capacity of well-known factor strategies—value, size, momentum, quality, and idiosyncratic volatility—by estimating the reduction in the factor risk premium for various transfers of capital.

We find that extremely large transfers of capital, which are several times larger than the current dollar allocations to the portfolios with the highest expected returns, are required to significantly decrease the value, momentum, and idiosyncratic factor premiums. In fact, for these strategies, an efficient allocation of capital which completely drives the factor premiums to zero, must take highly leveraged positions in value, small size, winner, high quality, and low volatility stocks. Thus, the model predicts that capacity in these strategies is large and even moving *all* capital from growth to value, loser to winner, junk to quality, or high volatility to stable stocks reduces the factor premiums by at most 50%. In most cases, the reduction in the factor premiums is much smaller. In contrast, small size strategies have the smallest equilibrium capacity and could diminish under large flows from large to small stocks.

Appendix: Completion Portfolios

Equation (3) written in "beta pricing language" of the CAPM is equivalent to the following "Euler equation" language of Hansen and Jagannathan (1991):

$$E(mR) = 1, \tag{A.1}$$

where m is the candidate pricing kernel and R is a gross return of a test asset. Equation (A.1) says that when evaluating the expectations of the return payoffs of the portfolios using the pricing kernel, we obtain the price of the portfolios today—which is $1 as we work in gross returns.

Hansen and Jagannathan (1991) show how to construct the pricing kernel that satisfies the Euler conditions in equation (A.1). The pricing kernel's volatility must be greater than a certain minimum threshold, and the pricing kernel with the minimum required volatility, m*, that solves equation (A.1) is given by

$$m^* = w^* \cdot R, \tag{A.2}$$

where the optimal weights, w*, on the test portfolios are given by

$$w^* = \frac{1}{R_f} + (R - \bar{R})'\Sigma_R^{-1}\left(\mathbf{1} - \frac{1}{R_f}\right) \cdot \tag{A.3}$$

In terms of computation, R is a $T \times N$ matrix of T observations of N asset gross returns; $\bar{R}$ is the average gross return of the portfolios; R_f is a given gross risk-free rate; and Σ_R^{-1} is the empirical covariance matrix of R. In practice, the pricing kernel tends to place large weights on portfolios with high excess returns and low weights on low excess return portfolios, adjusted for risk in the covariance matrix, Σ_R. Hansen and Jagannathan (1991) derive equation (A.3) as an OLS projection.

We compute the portfolio alphas using

$$\alpha = (\bar{R} - R_f) - \mathrm{E}^m(R - R_f), \tag{A.4}$$

where $\mathrm{E}^m(R - R_f)$ is the excess return implied by the choice of the pricing kernel m. We evaluate

$$\mathrm{E}^m(R - R_f) = \frac{cov(R,m)}{E(m)}, \tag{A.5}$$

which is the same as equation (3).

One way to derive the constants a and b for the CAPM in equation (4) is as follows. The constant a pins down the risk-free rate and solves $\mathrm{E}(m_{mkt}) = 1/R_f$. The constant b prices the market portfolio itself, $E(m_{mkt}R_{mkt}) = 1$.

When the optimal weights w* are normalized using $w^*/\boldsymbol{sum}(w^*)$, they represent the weights in the efficient portfolio (the tangency portfolio of the intersection of the Capital Allocation Line and the mean-variance frontier). In this case, the alphas in equation (5) are equal to zero.

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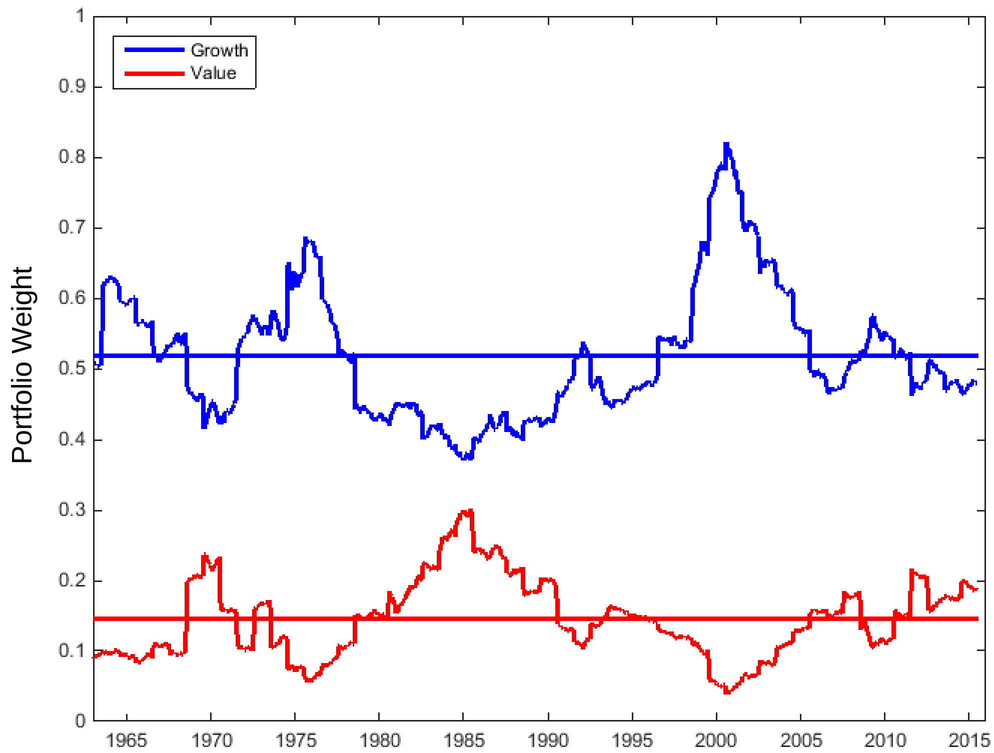
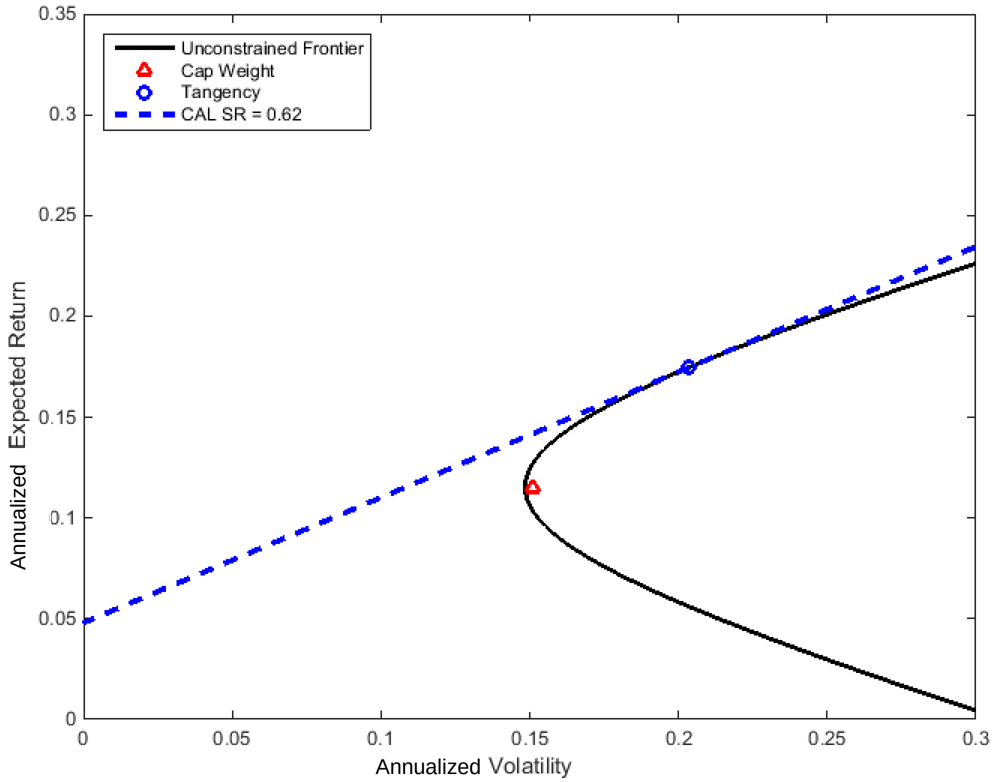
Exhibit 1[10]
Weights on Value and Growth



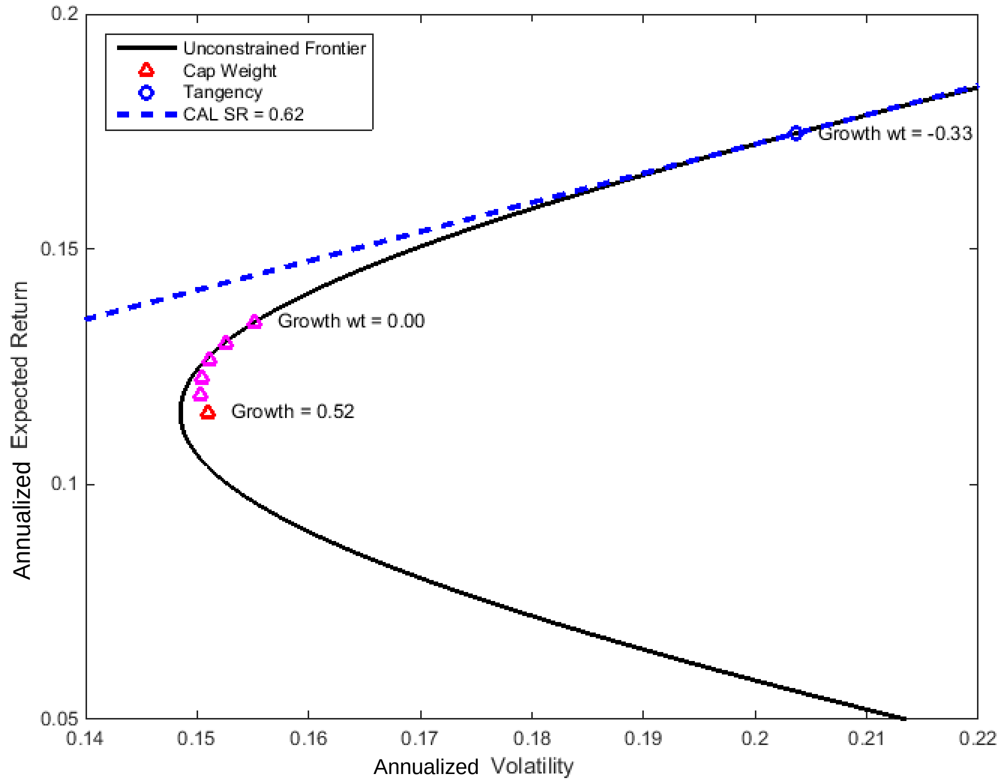
[10] Source: Fama French database from July 1963-December 2015. Data is available at http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

Exhibit 2[11]
Mean-Variance Frontier for Value-Growth



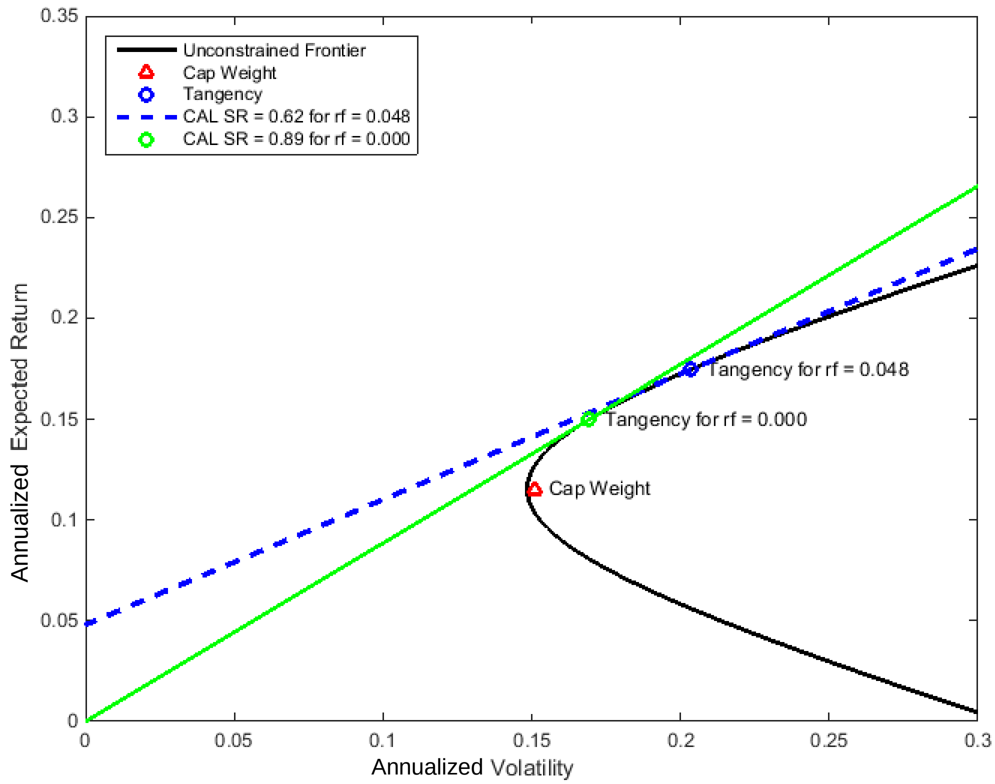
[11] CAL SR is represents Capital Allocation Line with Sharpe Ration
Source: Fama French database from July 1963-December 2015. Data is available at http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html
Annualized expected returns are stated in standard decimal notation.

Exhibit 3[12]
Mean-Variance Frontier for Value-Growth with Completion Portfolios



[12] Source: Fama French database from July 1963-December 2015. Data is available at http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html
Annualized expected returns are stated in standard decimal notation.

Exhibit 4[13]
Effect of Lower Risk-Free Rates on Value-Growth Efficient Frontier



[13] Source: Fama French database from July 1963-December 2015. Data is available at http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html
Annualized expected returns are stated in standard decimal notation.

Exhibit 5[14]
Factor Premiums Under Various Pricing Kernels

		Candidate Pricing Kernels					
		Market CAPM		Transfer Capital from Low Return to High Return Portfolio		Efficient Portfolio	
	Raw Return	Alpha	Weights	Alpha	Weights	Alpha	Weights
Value-Growth Portfolios							
Growth	5.80%	-1.18%	0.519	-1.71%	0.000	0.00%	-0.334
Neutral	6.92%	0.55%	0.336	-1.07%	0.336	0.00%	-0.746
Value	9.49%	2.96%	0.146	0.54%	0.664	0.00%	2.080
Value Premium		4.14%		2.25%		0.00%	
Size Portfolios							
Large	5.89%	-0.35%	0.809	0.63%	0.000	0.00%	-0.473
Neutral	8.74%	1.40%	0.139	0.88%	0.139	0.00%	2.109
Small	9.35%	1.73%	0.052	-0.14%	0.861	0.00%	-0.473
Size Premium		2.08%		-0.77%		0.00%	
Momentum Portfolios							
Losers	2.97%	-4.46%	0.214	-5.06%	0.000	0.00%	-0.682
Neutral	5.27%	-0.50%	0.447	-1.68%	0.447	0.00%	0.062
Winners	9.78%	3.47%	0.339	1.36%	0.553	0.00%	1.620
Momentum Premium		7.93%		6.42%		0.00%	
Profitability Portfolios							
Low Profitability	7.24%	-2.10%	0.167	-2.42%	0.000	0.00%	-1.107
Neutral	7.98%	-0.35%	0.385	-0.78%	0.385	0.00%	0.208
High Profitability	9.46%	1.08%	0.448	0.49%	0.615	0.00%	1.899
Quality Premium		3.18%		2.91%		0.00%	
Idiosyncratic Volatility Portfolios							
Volatile	8.47%	-4.28%	0.151	-4.47%	0.000	0.00%	-1.219
Neutral	10.45%	0.55%	0.303	-0.18%	0.303	0.00%	2.307
Stable	7.53%	0.88%	0.546	0.08%	0.697	0.00%	-0.088
Volatility Premium		5.16%		4.55%		0.00%	

[14] Source: Fama French database from July 1963-December 2015. Data is available at http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

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