Modeling Commodity Futures: Reduced Form vs. Structural Models

Pierre Collin-Dufresne
University of California - Berkeley
Presentation based on the following papers:

Stochastic Convenience Yield Implied from Interest Rates and Commodity Futures
forthcoming *The Journal of Finance*
joint with
Jaime Casassus
Pontificia Universidad Católica de Chile

Equilibrium Commodity Prices with Irreversible Investment and Non-Linear Technologies
joint with
Jaime Casassus
Pontificia Universidad Católica de Chile
Bryan Routledge
Carnegie Mellon University
Motivation

- Dramatic growth in commodity markets
  - trading volume, variety of contracts, number of underlying commodities

- Growth has been accompanied with high levels of volatility

- Commodity spot and futures prices exhibit empirical regularities ≠ financial securities (Fama and French (1987), Bessembinder et al. (1995))
  - Mean-reversion,
  - Convenience Yield,
  - ‘Samuelson effect.’

- Understanding the behavior of commodity prices important for:
  - Macroeconomic policies
  - Valuation of derivatives (short term)
  - Valuation and exercise of real options (long term)
Gold prices

- Stylized facts of spot and futures prices
  → Mean reversion (?), volatility
Gold futures prices

- Stylized facts of futures prices
  → Weak backwardation (?) & contango
  → Futures curve not uniquely determined by spot (non Markov)
  → Samuelson effect (?)
Crude oil prices

- Stylized facts of spot and futures prices
  → Mean reversion, heteroscedasticity, positive skewness (upward spikes)
Crude oil futures prices

- Stylized facts of futures prices
  - Strong (63%) & weak (83%) Backwardation & Contango
  - Non Markov spot price
  - Volatility of Futures prices decline with maturity (‘Samuelson effect’)
Short vs. long maturity futures

- Absence of arbitrage (in frictionless market) implies

\[ F(t, T) - S(t) = e^{r(t,T)-\delta(t,T)} S(t) \]

where \( r(t, T) \) is the interest rate and \( \delta(t, T) \) is the net convenience yield

- Convenience yield \( \approx \) ‘dividend’ accruing to holder of commodity (but not of futures)

- Time varying convenience yield (\( \delta(t, T) > 0 \)) necessary to explain backwardation (and possibly mean-reversion? Bessembinder et al. (95))
Expected spot vs. futures prices

- Gap between expected spot and futures price is a *risk premium*

\[ E_t[S(T) - S(t)] = (F(t, T) - S(t)) + \beta(t, T) \]

- Time-varying expected return (i.e., risk premium), \( \beta(t, T) \)
  can explain mean-reversion in spot if \( \beta(t, T) \uparrow \) when \( S_t \downarrow \). (Fama & French (88))

- Predictability of futures for expected spots?
Objectives

- Present a three-factor (‘maximal’) model for commodity prices that nests all Gaussian models
  (Brennan (91), Gibson Schwartz (90), Ross (97), Schwartz (97), Schwartz and Smith (00))

- Study stylized facts of commodity spot and futures prices

- Examine sources of mean-reversion in commodity prices
  - maximal convenience yield vs. time varying risk premia

- Test to what extent the restrictions in existing models are binding

- Illustrate economic significance of maximal model
  - option pricing vs. risk management decisions

- Compare commodities of different nature
  - productive assets: crude oil and copper
  - financial assets: gold and silver
Main results for reduced-form model

→ Three-factors are necessary to explain dynamics of commodity prices

→ In the maximal model the convenience yield is a function of the spot price, interests rates and an idiosyncratic factor

→ Convenience yields are positive and increasing in price level and interest rates (in particular for crude oil and copper)

→ Convenience yields are economically significant for derivative pricing

→ Time-varying risk premium seem more significant for store-of-value assets

→ Risk premia of prices is decreasing in the price level (counter-cyclical)

→ Economically significant implications for risk management (VAR)
Maximal model for commodity prices

- *Maximal*: most general (within certain class) model that is econometrically identified

- Canonical representation of a three-factor Gaussian model for spot prices
  \[
  X(t) := \log S(t) = \phi_0 + \phi_Y^T Y(t)
  \]

- \(Y(t)\) is a vector of three latent variables
  \[
  dY(t) = -\kappa^Q Y(t) dt + dZ^Q(t)
  \]

- Maximality implies
  - \(\kappa^Q\) is a lower triangular matrix
  - \(dZ^Q\) is a vector of independent Brownian motions

- Futures prices observed for all maturities, obtained in closed-form (Langetieg 80):
  \[
  F^T(t) = E_t^Q \left[ e^{X(T)} \right]
  \]
Interest rates and convenience yields

- Interest rates follow a one-factor process
  \[ r(t) = \psi_0 + \psi_1 Y_1(t) \]

- Bond prices observed across maturities obtained in closed-form (Vasicek 77):
  \[ P_T(t) = E_t^Q[e^{-\int_t^T r(s)ds}] \]

- Absence of arbitrage implies that (this defines the convenience yield!):
  \[ E_t^Q[dS(t)] = (r(t) - \delta(t))S(t)dt \]

- The implied convenience yield in the maximal model is affine in \( Y(t) \)
  \[ \delta(t) = \psi_0 - \frac{1}{2} \phi_Y^\top \phi_Y + \psi_1 Y_1(t) + \phi_Y^\top \kappa^Q Y(t) \]
Economic representation of the maximal model

- The ‘maximal’ model is

  \[ \delta(t) = \widehat{\delta}(t) + \alpha_X X(t) + \alpha_r r(t) \]

  \[ dX(t) = \left( r(t) - \delta(t) - \frac{1}{2} \sigma_X^2 \right) dt + \sigma_X dZ_X^Q(t) \]

  \[ d\widehat{\delta}(t) = \kappa_{\widehat{\delta}}^Q \left( \theta_{\widehat{\delta}}^Q - \widehat{\delta}(t) \right) dt + \sigma_{\widehat{\delta}} dZ_{\widehat{\delta}}^Q(t) \]

  \[ dr(t) = \kappa_r^Q \left( \theta_r^Q - r(t) \right) dt + \sigma_r dZ_r^Q(t) \]

  and the Brownian motions are correlated.

- The maximal convenience yield model nests most models in the literature
  e.g. \( \alpha_X = \alpha_r = 0 \Rightarrow \) three-factor model of Schwartz (1997)

- \( \alpha_X > 0 \): mean-reversion in prices under the risk-neutral measure (Samuelson effect)
  - consistent with futures data (empirical) and with ‘Theory of Storage’ models (theoretical)

- \( \alpha_r \): convenience yield may depend on interest rates
  - if holding inventories becomes costly with high interest rates then \( \alpha_r > 0 \)
Specification of risk premia necessary for estimation

- Risk-premia is a linear function of states variables (Duffee (2002))

\[
\beta(t) = \begin{pmatrix} \beta_{0r} \\ \beta_{0\tilde{\delta}} \\ \beta_{0X} \end{pmatrix} + \begin{pmatrix} \beta_{rr} & 0 & 0 \\ 0 & \beta_{\tilde{\delta}\tilde{\delta}} & 0 \\ \beta_{Xr} & \beta_{X\tilde{\delta}} & \beta_{XX} \end{pmatrix} \begin{pmatrix} r(t) \\ \tilde{\delta}(t) \\ X(t) \end{pmatrix}
\]

and

\[
dZ^Q(t) = \sigma^{-1} \beta(t) dt + dZ^P(t)
\]

- Time-varying risk-premia is another source of mean-reversion under historical measure

  → Mean-reversion in commodity prices: \( \kappa_P^X = \alpha_X - \beta_{XX} \)

  → Mean-reversion in convenience yield: \( \kappa_P^{\tilde{\delta}} = \kappa_Q^{\tilde{\delta}} - \beta_{\tilde{\delta}\tilde{\delta}} \)

  → Mean-reversion in interest rates: \( \kappa_P^r = \kappa_Q^r - \beta_{rr} \)
Data and empirical methodology

- Weekly data of futures contracts on crude oil, copper, gold and silver
  - Jan-1990 to Aug-2003
  - with maturities $\{1,3,6,9,12,15,18\}$ months + some longer contracts

- Build zero-coupon bonds for same period of time
  - with maturities $\{0.5,1,2,3,5,7,10\}$ years

- Maximum likelihood estimation with time-series and cross-sectional data
  - state variables $\{r, \hat{\delta}, X\}$ are not directly observed, but futures prices and bonds are observed
  - assume some linear combination of futures and bonds to be observed without error
  - invert for the state variables from observed data
  - first two Principal Components of futures curve are perfectly observed
  - first Principal Component of term structure of interest rate is perfectly observed
  - remaining PCs are observed with errors that follow AR(1) process
Empirical results: sources of mean-reversion

- **Convenience yields**
  - $\alpha_X$ is significant and positive, and highest for Oil and lowest for Gold
  - $\alpha_r$ is significant and positive for Oil and Gold

- **Maximum-likelihood parameter estimates for the model**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Gold Estimate (Std. Error)</th>
<th>Crude Oil Estimate (Std. Error)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_X$</td>
<td>0.000 (0.000)</td>
<td>0.248 (0.010)</td>
</tr>
<tr>
<td>$\alpha_r$</td>
<td>0.332 (0.046)</td>
<td>1.764 (0.083)</td>
</tr>
</tbody>
</table>

- **Likelihood ratio test** ($\text{Prob}\{\chi^2 \geq 5.99\} = 0.05$)

<table>
<thead>
<tr>
<th>Restriction</th>
<th>Gold</th>
<th>Crude Oil</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_r = \alpha_X = 0$</td>
<td>5.60</td>
<td>1047.20</td>
</tr>
</tbody>
</table>

Q-Group Seminar - Ocean Reef

Spring 2005
Empirical results: sources of mean-reversion

- Unconditional moments (convenience yield)

<table>
<thead>
<tr>
<th>Unconditional Moments</th>
<th>Gold</th>
<th>Crude Oil</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[\delta]$</td>
<td>0.009</td>
<td>0.109</td>
</tr>
<tr>
<td>Stdev ($\delta$)</td>
<td>0.010</td>
<td>0.210</td>
</tr>
</tbody>
</table>
Empirical results: sources of mean-reversion

- **Time-varying risk premia**
  - For metals most risk-premia coefficients associated with prices are significant
  - $\beta_{XX}$ is always negative, higher mean-reversion under historical measure

- **Maximum-likelihood parameter estimates for the model**

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<tr>
<th>Parameter</th>
<th>Gold Estimate (Std. Error)</th>
<th>Crude Oil Estimate (Std. Error)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_{0X}$</td>
<td>1.858 (1.539)</td>
<td>1.711 (0.964)</td>
</tr>
<tr>
<td>$\beta_{Xr}$</td>
<td>-2.857 (2.452)</td>
<td></td>
</tr>
<tr>
<td>$\beta_{XX}$</td>
<td>-0.301 (0.271)</td>
<td>-0.498 (0.313)</td>
</tr>
</tbody>
</table>

- **Likelihood ratio test** (Prob$\{\chi^2 \geq 11.07\} = 0.05$ and Prob$\{\chi^2 \geq 14.07\} = 0.05$)

<table>
<thead>
<tr>
<th>Restriction</th>
<th>Gold</th>
<th>Crude Oil</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_{1Y} = 0$</td>
<td>20.01</td>
<td>13.16</td>
</tr>
<tr>
<td>$\alpha_r = \alpha_X = 0$ and $\beta_{1Y} = 0$</td>
<td>23.96</td>
<td>1057.92</td>
</tr>
</tbody>
</table>
Two-year maturity European call option

- The strike prices are
  - $350 per troy ounce for the option on gold
  - $25 per barrel for the option on crude oil

- Ignoring maximal convenience yield induces overestimation of call option values
  - mean-reversion (under $Q$) reduces term volatility which decreases option prices
  - convenience yield acts as a stochastic dividend (which increases with $S_t$)
Value at Risk

- Distribution of returns and VAR for holding the commodity for 5 years
- VAR calculated from the *total* return at 5% significance level
- Ignoring time-varying risk-premia induces overestimation of VAR
  - mean-reversion reduces term volatility which decreases VAR
Conclusions from reduced-form model

- Propose a maximal affine model for commodity prices
  - convenience yield and risk-premia are affine in the state variables
  - disentangles two sources of mean-reversion in prices
  - nests most existing B&S type models
    (Brennan (91), Gibson Schwartz (90), Ross (97) Schwartz (97), Schwartz and Smith (00))

- Three factors are necessary to explain dynamics of commodity prices

- Maximal convenience yield mainly driven by spot price
  - is highly significant for assets used as input to production (i.e. Oil)
  - explains strong backwardation in commodities
  - is economically significant for derivative pricing on productive assets

- Time-varying risk-premia
  - are more significant for store-of-value assets
  - risk premium of commodity prices is decreasing in the price level
  - are economically significant for risk management decisions

- Robust to allowing for jumps in spot dynamics (small impact on futures)
Potential Issues with Reduced-Form Approach

- **Reduced-form model:**
  - Exogenous specification of spot price process, convenience yield, and interest rate.
  - Arbitrage pricing of Futures contracts.
  - Financial engineering (Black & Scholes) data-driven approach.

- **Structural Model useful benchmark to design reduced-form model:**
  - endogenous modeling of Convenience Yield.
  - helpful for long horizon decisions (only short term futures data available).
  - provide theoretical foundations for reduced-form dynamics.
  - avoid data-mining, over-parametrization?
Existing Theories of Convenience Yield

• ‘Theory of Storage’: (Kaldor (1939), Working (1948), Brennan (1958))
  – Why are inventories high when futures prices are below the spot?
  – Inventories are valuable because help smooth demand/supply shocks.
  – Used by the Reduced-form literature to justify informally ‘dividend.’

• – ‘Stockout’ literature (Deaton and Laroque (1992), Routledge, Seppi and Spatt (2000) (RSS))
  – Competitive rational expectation models with risk neutral agents.
  – Stockouts (i.e., non-negativity constraint on inventories) explain ‘Backwardation.’
  – Inconsistent with frequency of backwardation in data?

• ‘Option’ approach (Litzenberger and Rabinowitz (1995))
  – Oil in the ground as a call option on oil price with strike equal to extraction cost.
  – Convenience yield must exist in equilibrium for producers to extract (i.e., exercise their call).
  – Predicts Backwardation 100% of the time (flexibility of production technology?).

• ‘Technology’ approach (Sundaresan and Richard (1978))
  – Convenience yield is similar to a real interest on foreign currency (where commodity is numeraire).
A Structural Model

• Equilibrium Model of a Commodity - Input to Production:
  – Oil is produced by oil wells with variable flow rate (adjustment costs).
  – Investment in new oil wells is costly (fixed and variable costs).
  – Single consumption good produced with two inputs: Oil and Consumption good.

• Main results:
  – Mean-reverting, heteroscedastic, positively skewed prices.
  – Price is non-Markov (regime switching): depends on ‘distance to investment.’
  – Price can exceed its marginal production costs (fixed costs).
  – Generates Backwardation at observed frequencies.
  – Convenience yields arises endogenously (adjustment costs).

• Empirical ‘implementation:’
  – Estimation is consistent with the predictions from structural model.
A general equilibrium model for commodity prices
Representative Agent in a Two-sector economy

The RA owns the technologies of sectors \( Q \) and \( K \) and maximizes

\[
J(K, Q) = \sup_{\{C_t, X_t, dI_t\}} E_0 \left[ \int_0^\infty e^{-\rho t} U(C_t) dt \right]
\]

- \( X_t \): How much to invest in commodity sector
- \( I_t \): When to invest in commodities
- Capital stock:
  \[
dK_t = (f(K_t, \bar{i}Q_t) - C_t) dt + \sigma K_t db_t - \beta(K_t, Q_t, X_t)dI_t
\]
- Commodity stock:
  \[
dQ_t = -(\bar{i} + \delta)Q_t dt + X_t dI_t
\]
- Flow rate \( \bar{i} \) is fixed (\( \infty \) adjustment cost - relaxed below).
- Irreversible investment with increasing returns to scale (fixed costs)
  \[
  \beta(K_t, Q_t, X_t) = \beta_K K_t + \beta_Q Q_t + \beta_X X_t
  \]
  \implies Investment in commodity sector is intermittent and lumpy
Solution using standard Dynamic programming

- If investment is perfectly reversible \((X_t \leq 0\) possible, no fixed costs: \(\beta_K = \beta_Q = 0\) and \(\beta_X > 0\)) then
  
  - The optimal policy is simply to keep a constant \(Q/K\) ratio:
  
  \[
  \frac{Q^*_t}{K^*_t} = \left( \frac{\alpha i^n \eta}{\beta_X (\hat{i} + \delta)} \right)^{\frac{1}{1-n}}
  \]

  - The oil price is simply \(S_t = \beta_X\).

- If investment is irreversible then the optimal policy is discrete and ‘lumpy’
  
  - **no-investment** region: \(J(K_t - \beta_t, Q_t + X_t) < J(K_t, Q_t)\)
  
  - **investment** region: \(J(K_t - \beta_t, Q_t + X_t) \geq J(K_t, Q_t)\)

- Under some technical condition can solve for the HJB equation for optimal policy.
Simulation for state variable $z_t = \log\left(\frac{Q_t}{K_t}\right)$

- Regulated dynamics at the investment boundary
- $dz_t = \mu z_t dt - \sigma db_t + \Lambda_z dI_t$ where $\Lambda_z = z_2 - z_1$ if $dI_t = 1$
Equilibrium Asset Prices

• In equilibrium, financial assets are characterized by:

\[ \xi_t = e^{-\rho t} \frac{J_K(K_t, Q_t)}{J_K(K_0, Q_0)} \]
\[ r_t = f_K(K_t, \tilde{Q}_t) - \sigma \lambda_t \]
\[ \lambda_t = -\sigma \frac{K_t J_{KK}}{J_K} \]
\[ \Lambda_B = -\frac{\beta_K}{1 - \beta_K} \]

• Any financial claim satisfies:

\[ \frac{dH_t}{H_t} = \mu_{Ht} dt + \sigma_{Ht} db_t + \Lambda_B dI_t \]

• Subject to the equilibrium conditions

\[ \mu_{Ht} = r_t + \lambda_t \sigma_{Ht} \quad \text{(1)} \]

⇒ All financial securities jump by fixed amount at investment date (wealth effect).
Commodity price

- The equilibrium commodity price is the transfer price from sector $Q$ to sector $K$, i.e. the representative agent’s shadow price for that unit

$$J(K_t, Q_t) = J(K_t + S_t \epsilon, Q_t - \epsilon) \text{ or } S_t = \frac{J_Q}{J_K}$$

- Dynamics of the commodity price process

$$\frac{dS_t}{S_t} = \mu_S dt + \sigma_S db_t + \Lambda_S dI_t$$

- From first order conditions and stochastic discount factor

$$\Lambda_S = \frac{\beta_Q - \beta_K \beta_X}{1 - \beta_K}$$

- Note $\Lambda_S \neq \Lambda_B$, but does not imply arbitrage (!)
Commodity prices as a function of the state variable

- Two opposite forces: demand / depreciation vs. investment probability
- The spot price process itself is not a Markov process
- The spot price follows a two-regime process: $\varepsilon_t = \begin{cases} 1 & \text{if } z > z_{S_{\text{max}}} \\ 2 & \text{if } z_{1} < z \leq z_{S_{\text{max}}} \end{cases}$
Commodity price process from the equilibrium model

- Regime switching model \((\varepsilon_t = 1, 2)\)

\[
\frac{dS_t}{S_t} = \mu_s(S_t, \varepsilon_t) dt + \sigma_s(S_t, \varepsilon_t) dB_t + \Lambda_{S_t} dI_t
\]

- Predictions about the dynamics of the commodity price process
Simulation of commodity prices

- Price can be well above its marginal production cost ($\beta_x$)
- Mean-reversion.
Futures prices

- The stochastic process for the futures prices $H(z_t, T)$ is

$$\frac{dH_t}{H_t} = \mu_{Ht}dt + \sigma_{Ht}db_t + \Lambda_{Ht}dI_t$$

subject to the equilibrium conditions

$$\frac{\mu_{Ht}}{\sigma_{Ht}} = \lambda_t = \text{market price of risk} \text{ and } H(z_1, t) = H(z_2, t)$$

- The futures price satisfies the following PDE

$$\frac{1}{2}\sigma^2 H_{zz} + (\mu_z - \sigma \Lambda_b)H_z - H_t = 0$$

and boundary condition $H(z_t, 0) = S(z_t)$
Futures prices on the commodity for different maturities
Net convenience yield

- $y_t = \frac{\dot{i}}{S_t} \left( f_q(K_t, \dot{\bar{Q}}_t) - S_t \right) - \delta$
Risk premium for commodity prices

- Risk premium is: \( \sigma_{St} \lambda_t = \gamma \text{cov} \left( \frac{dS_t}{S_t}, dC_t \right) \)

\[
\frac{dS_t}{S_t} = (r_t - y_t + \sigma_{St} \lambda_t) dt + \sigma_{St} dw + \Lambda_{St} dI_t
\]

- consistent with empirical findings
Calibration of Model Parameters

- Fix $\eta = 0.04$ consistent with recent RBC studies (Finn (00), Wei (03))
- Fix $\delta, \rho$ to reasonable numbers (for identification).
- Estimate $\alpha, \bar{i}, \sigma, \gamma$ and costs $\beta_X, \beta_K, \beta_Q$ to fit a few moments:
  1. US annual Oil Consumption /US GDP $\approx \bar{i}QS/f(K, \bar{i}Q)$.
  2. US Consumption (non-durables + services) / US GDP $\approx C/f(K, \bar{i}Q)$
  4. Data averages for Cons/GDP from 49-02 and from 97 to 03 for Futures.
  5. Model averages estimated by simulating stationary distribution of $z$.

<table>
<thead>
<tr>
<th>Production technologies</th>
<th></th>
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<tbody>
<tr>
<td>Productivity of capital $K$, $\alpha$</td>
<td>0.23</td>
</tr>
<tr>
<td>Oil share of output, $\eta$</td>
<td>0.04</td>
</tr>
<tr>
<td>Demand rate for oil, $\bar{i}$</td>
<td>0.07</td>
</tr>
<tr>
<td>Volatility of return on capital, $\sigma$</td>
<td>0.263</td>
</tr>
<tr>
<td>Depreciation of oil, $\delta$</td>
<td>0.02</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Irreversible investment</th>
<th></th>
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</thead>
<tbody>
<tr>
<td>Fixed cost ($K$ component), $\beta_K$</td>
<td>0.016</td>
</tr>
<tr>
<td>Fixed cost ($Q$ component), $\beta_Q$</td>
<td>0.272</td>
</tr>
<tr>
<td>Marginal cost of oil, $\beta_X$</td>
<td>17</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Agents preferences</th>
<th></th>
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</thead>
<tbody>
<tr>
<td>Patience, $\rho$</td>
<td>0.05</td>
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<tr>
<td>Risk aversion, $\gamma$</td>
<td>1.8</td>
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<tr>
<td></td>
<td>Historical data</td>
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<tr>
<td>-----------------------</td>
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<tr>
<td></td>
<td>Mean</td>
</tr>
<tr>
<td><strong>Crude oil futures prices (US$/bbl)</strong></td>
<td></td>
</tr>
<tr>
<td>1-months contract</td>
<td>23.62</td>
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<tr>
<td>6-months contract</td>
<td>22.61</td>
</tr>
<tr>
<td>12-months contract</td>
<td>21.65</td>
</tr>
<tr>
<td>18-months contract</td>
<td>21.05</td>
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<tr>
<td>24-months contract</td>
<td>20.71</td>
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<tr>
<td>36-months contract</td>
<td>20.42</td>
</tr>
<tr>
<td>48-months contract</td>
<td>20.28</td>
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<tr>
<td>60-months contract</td>
<td>20.23</td>
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<tr>
<td>72-months contract</td>
<td>20.42</td>
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<tr>
<td><strong>Macroeconomic ratios</strong></td>
<td></td>
</tr>
<tr>
<td>consumption of oil-output ratio</td>
<td>2.16%</td>
</tr>
<tr>
<td>output-consumption of capital ratio</td>
<td>1.8</td>
</tr>
</tbody>
</table>

![Graph of Mean of Futures vs Maturity](image1.png)

![Graph of Vol of Futures vs Maturity](image2.png)
Reduced-Form model with two regimes

- Estimate a Reduced Form model that is consistent with the equilibrium model

\[ dS_t = \mu_S(S_t, \varepsilon_t)S_t dt + \sigma_S(S_t, \varepsilon_t)S_t db_t \]

where

\[ \mu_S(S_t, \varepsilon_t) = \alpha + \kappa \varepsilon (\log[S_{Max}] - \log[S_t]) \]
\[ \sigma_S(S_t, \varepsilon_t) = \sigma \varepsilon \sqrt{\log[S_{Max}] - \log[S_t]} \]

and \( \varepsilon_t \) is a two-state Markov chain with transition (Poisson) probabilities

\[
P_t = \begin{bmatrix}
1 - \lambda_1 dt & \lambda_1 dt \\
\lambda_2 dt & 1 - \lambda_2 dt
\end{bmatrix}
\]

- Define \( \varepsilon_t = \begin{cases} 
1 & \text{in the far-from-investment region} \\
2 & \text{in the near-investment region} 
\end{cases} \)
Estimation and predictions

- Maximum Likelihood (weekly crude oil prices from 1/1982 to 8/2003)
- Estimate $\Theta = \{\alpha, \kappa_1, \kappa_2, \sigma_1, \sigma_2, S_{Max}, \lambda_1, \lambda_2\}$

<table>
<thead>
<tr>
<th>far-from-investment state</th>
<th>near-investment state</th>
<th>Common parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter</td>
<td>Estimate</td>
<td>t-ratio</td>
</tr>
<tr>
<td>$\lambda_1$</td>
<td>0.984</td>
<td>3.2</td>
</tr>
<tr>
<td>$1/\lambda_1$</td>
<td>1.017</td>
<td>3.2</td>
</tr>
<tr>
<td>$\lambda_2/(\lambda_1 + \lambda_2)$</td>
<td>83.7%</td>
<td></td>
</tr>
<tr>
<td>$\kappa_1$</td>
<td>0.319</td>
<td>2.7</td>
</tr>
<tr>
<td>$\sigma_1$</td>
<td>0.251</td>
<td>25.8</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>-0.248</td>
<td>-2.4</td>
</tr>
<tr>
<td>$S_{Max}$</td>
<td>39.797</td>
<td>99.7</td>
</tr>
<tr>
<td>$\lambda_1$</td>
<td>$\ll \lambda_2$</td>
<td></td>
</tr>
</tbody>
</table>

- Predictions from the structural model

\[
\mu_S(S_{Max}, \varepsilon_t) < 0 \iff \alpha < 0 \\
\mu_S(0, 1) > 0 \text{ and } \kappa_1 > 0 \\
\mu_S(S_t, 2) < 0 \text{ and } \kappa_2 < 0 \\
\lambda_1 \ll \lambda_2
\]
Regime-switching estimation of commodity price process

- Regime-switching model

\[ dS_t = (\alpha + \kappa \varepsilon (\log[S_{Max}] - \log[S_t]))S_t dt + \sigma \varepsilon \sqrt{\log[S_{Max}] - \log[S_t]} S_t dB_t \]

- Estimated drift and volatility of crude oil returns

![Graph showing Drift and Volatility vs Crude Oil Price](image-url)
Smoothed inferences for the regime switching model

- Historical crude oil price
- Inferred probability of \textit{near-investment} state
Conclusion

• Structural model of commodity whose primary use is as an input to production.

• Infrequent lumpy investment in commodity determines two regimes for the commodity price, depending on the distance to the investment trigger.

• The spot price exhibits mean reversion, heteroscedasticity, and regime switching.

• Convenience yield has two endogenous components which arise because the commodity helps smooth production in response to demand/supply shocks.

• The model can generate the frequency of backwardation observed in the data.

• Estimates of a reduced-form regime switching model seem consistent with the predictions of the model.

• Future work:
  – Estimation/Calibration of parameters based on data.
  – Implication of model prediction for reduced-form modeling, pricing and hedging of options.