Market Microstructure Invariants

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Motivation

It is important for asset managers to understand

- The level of **transaction costs**:
  - **market impact** (increasing cost per share traded),
  - **bid-ask spread** (fixed cost pre share traded);
- How transactions costs vary cross-sectionally across stocks as level of **trading activity** varies.

Motivation

Understanding **level** of **transaction costs** helps answer some important questions:

- What percentage of "alpha" is lost due to transaction costs?
- How much money can be allocated to a seemingly profitable strategy before it becomes non-economical due to high transaction costs?

Motivation

Understanding **cross-sectional variation** in **transaction costs** helps answer the following questions:

- Is it reasonable to restrict the rate of trading to a fixed percentage of trading volume, say 1% of daily volume for all stocks, or should the maximum percentage of average daily volume vary across stocks?

- If one broker executes orders for small stocks and another broker executes orders for large stocks, how can we compare their performance?

Overview

Our goal is to explain how **market impact** and **bid-ask spread** vary across stocks with different **trading activity**.

- We develop a **model of market microstructure invariants** that generates predictions concerning cross-sectional variations of these variables.
- These predictions are tested using a data set of portfolio transitions and find a strong support in the data.
- The model implies a **simple formula** for market impact and bid-ask spread as functions of observable dollar trading volume and volatility, which provides answers for above questions.

A Framework

When portfolio managers trade stocks, they can be thought of as playing **trading games**. Since managers trade many different stocks, we can think of them as playing many different trading games simultaneously, a different game for each stock.

The intuition behind a trading game was first described by Jack Treynor (1971) and later formalized by Albert S. Kyle (1985). We use an approach similar to Kyle (1985), but other market microstructure models would lead to similar results.

A Trading Game

Informed trader knows the terminal value of the asset and optimally trades against liquidity (noise) traders as well as a risk-neutral market maker:

- **Informed trader** makes a profit trading as monopolist on private information.
- **Liquidity traders** make losses from trading without information, because anonymous market cannot distinguish them from informed traders.
- **Market makers** are competitive and break even, since losses trading with informed equal gains trading with noise traders.

Market Impact

Market impact results from adverse selection due to trading on private information.

Market impact is ratio of **"informed trading"** to **"noise trading"**:

- More **informed trading** implies proportionally more price impact, i.e., less depth.
- More **noise trading** implies proportionally less price impact, i.e., more depth.

Market impact is permanent.

Properties of Dynamic Equilibrium

- Market impact parameter is constant over time (inter-temporal market depth arbitrage);
- Market impact is linear in trade size;
- Informed trader smoothes out trading over time;
- Price volatility is constant over time;
- Prices have martingale property, implying volatility of prices equals rate at which information arrives.

Bid-Ask Spread

We can modify the theory for predictions about **bid-ask spread**:

- If market makers are **imperfectly** competitive, transitory price impact is greater than permanent price impact.
- Transitory price impact is related to profits by market makers.
- The impact is proportional to size of orders, but thinking of this as a fixed spread might be reasonable, since large orders may be broken into many small pieces.

Games Across Stocks

Stocks are different in terms of their **trading activity**: dollar trading volume, volatility etc. Trading games look different across stocks at first sight!

Our intuition is that trading games are the **same across stocks**, except for the **length of time** over which these games are played or the **speed** with which they are played. The underlying parameters remain invariant across stocks with different levels of trading activity.

Games Across Stocks

Only the **speed** with which time passes **varies**, when trading activity varies:

- **For active stocks** (high trading volume and high volatility), trading games are played at a **fast pace**, i.e. the length of trading day is small.
- **For inactive stocks** (low trading volume and low volatility), trading games are played at a **slow pace**, i.e. the length of trading day is large.

The length of a trading day is related to **market efficiency**. The shorter is the trading day, the more efficient is the market.

Games Across Stocks

The underlying parameters that describe the game do not change, when trading activity varies. The following **"deep" parameters** remain **constant** across stocks, per **"trading game"**:

- number of noise traders,
- sizes of risks ("bets") taken by noise traders,
- size of risks taken by the informed trader,
- profitability of private information,
- quality of informed trade's information (cost of acquiring information).

Our model focusses on the constant **number of noise traders** and the constant **sizes of their bets**.

More formally...

One "trading day" corresponds to H **calendar days**.

The **market impact** is

$$\lambda = \frac{\sigma_{V,H}}{\sigma_{U,H}}$$

where

- $\sigma_{V,H}$ is the standard deviation of **private information** observed by informed trader.
- $\sigma_{U,H}$ is the standard deviation of changes in the **inventory of noise traders**.

Parameter $\sigma_{V,H}$

Parameter $\sigma_{V,H}$ is the standard deviation of **private information** observed by informed trader. For a trading day (H calendar days),

$$\sigma_{V,H} = \sigma_{V,1} \times \sqrt{H}$$

where $\sigma_{V,1}$ is readily estimated from daily **volatility** and **prices**:

$$\sigma_{V,1} = \sigma_r \times P.$$

Parameter $\sigma_{U,H}$

Parameter $\sigma_{U,H}$ is the standard deviation of changes in the **inventory of noise traders**. For a trading day (H calendar days),

$$\sigma_{U,H} = \sigma_{U,1} \times \sqrt{H}$$

Parameter $\sigma_{U,1}$ corresponds to daily **order imbalances**. Order imbalances depend on the **composition** of **trading volume**.

- Many small independent noise trades generate large trading volume but small order imbalances.
- In Kyle(1985), liquidity trading follows a Brownian motion process (infinitely many small independent noise trades), which implies that expected trading volume is infinite.
- A few large independent noise trades generate small trading volume but large order imbalances.

Parameter $\sigma_{U,H}$

We approximate the Brownian motion with a **compound Poisson process** with **trade arrival rate** γ_1 and **trade size** a random variable $\tilde{Q}$ with $\bar{Q} = E|\tilde{Q}|$ and the standard deviation $\sigma_Q = \theta \bar{Q}$. $\bar{Q}$ and γ_1 vary across stocks. Daily trading volume $V = \gamma_1 \times \bar{Q}$.

$$\sigma_{U,H} = \sigma_{U,1} \times \sqrt{H}$$

where $\sigma_{U,1} = \sigma_Q \times \gamma_1^{1/2} = \theta \bar{Q} \gamma_1^{1/2}$.

Key intuition:

- **Trading volume** is proportional to γ_1.
- **Order imbalances** are proportional to $\gamma_1^{1/2}$.

Bet Size

We think of **liquidity trades** as bets whose size is measured by dollar standard deviation over time.

Bet size over **a calendar day (1 day)**:

$$B_1 = \sigma_r \times P \times \sigma_Q$$

Bet size over **a trading day (H days)**:

$$B_H = B_1 \times \sqrt{H}$$

Bet Frequency

Bets arrive in the market with an assumed **frequency**.

Bet frequency per **calendar day (1 day)**:

$$\gamma_1$$

Bet frequency per **trading day (H days)**:

$$\gamma_H = \gamma_1 \times H$$

Theories of Market Microstructure Invariants

We propose three theories:

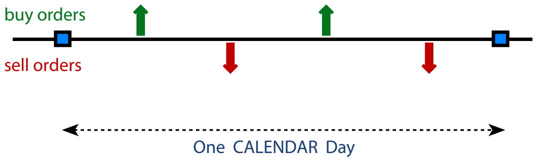
1. Our preferred **Theory of "Trading Game Invariance:"** based on the intuition that deep parameters of the **trading game itself** are invariant (both number of bets and their size per trading game), but the length of the trading games varies across stocks.

2. Naive alternative **Theory of "Invariant Bet Frequency"** per calendar day: based on intuition that more volume results from larger liquidity trades but not from more liquidity trades, per calendar day.

3. Naive alternative **Theory of "Invariant Bet Size**: based on the intuition that more volume results from more liquidity trades but not from larger liquidity trades, per calendar day.

A Benchmark Stock

Benchmark Stock - daily volatility $\sigma^* = 200$ bps, price $P^* = \$40$, volume $V^* = 1$ million shares. Liquidity trades over a calendar day:



Arrival Rate $\gamma^* = 4$

Avg. Order Size $\bar{Q}^*$ **as fraction of** $V^* = 1/4$

Market Impact of 1/4 $V^* = 200$ bps / $4^{1/2} = 100$ bps

1. Trading Game Invariance - Assumptions

We assume that the following underlying parameters remain constant as the level of **trading activity** varies:

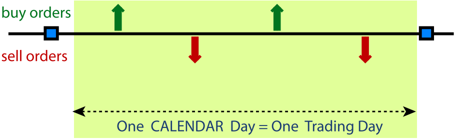
- **The number of bets** per "trading game" γ_H is **constant**.
- **The risk of a bet** per "trading game" B_H is **constant**.

The model implies that when **trading activity** increases by **one percent**:

- Size of a bet σ_Q increases by $1/3$ of one percent.
- Frequency of bets γ_1 per calendar day increases by $2/3$ of one percent.
- Length of trading game H decreases by $2/3$ of one percent.

1. Trading Game Invariance - Intuition

Benchmark Stock with Volume V^*

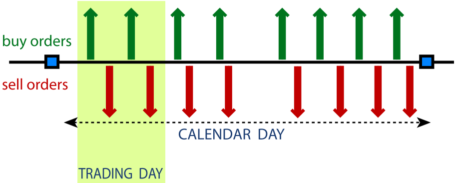


Avg. Order Size $\bar{Q}^*$ as fraction of V^* $= 1/4$

Market Impact of $1/4\ V^*$

$= 200$ bps $/\ 4^{1/2} = 100$ bps

Stock with Volume $=$ 8 $\times\ V^*$



Avg. Order Size $\bar{Q}$ as fraction of V

$= 1/16 = 1/4 \times 8^{-2/3}$

Market Impact of $1/16\ V$

$= 200$ bps $/\ 16^{1/2} = 50$ bps

Market Impact of $1/4\ V$

$= 4 \times 50$ bps $= 100$ bps $\times 8^{1/3}$

1. Trading Game Invariance - Intuition

- **Number of bets** per "trading game" γ_H is **constant**:
  - Frequency of bets per day increases by 2/3 of one percent.
  - Length of trading game decreases by 2/3 of one percent.
  - Increased frequency exactly offsets shorter length of game.

- **Standard deviation of bet risk** over life of bet B_H is **constant**:
  - Larger size of bet increases bet risk by 1/3 of one percent.
  - Shorter length of game decreases variance by 2/3 of one percent, i.e., standard deviation by 1/3 of one percent.
  - Increased size of bet exactly offsets decreased risk.

1. Trading Game Invariance - Predictions

If **trading activity** increases by **one percent**, some algebra implies the following cross-sectional predictions:

- **Average size of a liquidity trade**, as a percentage of average daily volume, decreases by **2/3 of one percent**;
- **Market impact** of trading X percent of average daily volume increases by **1/3 of one percent**;
- **Bid-ask spread** decreases by **1/3 of one percent**.

1. Trading Game Invariance - Prediction Math

Define **"Trading Activity"** W as product of dollar volume and volatility σ_r, generating a measure of gross risk transfer:

$$W = V \times P \times \sigma_r.$$

The Model of Trading Game Invariance implies:

- **Market Impact**: $\lambda_{TG} = \theta^{-1}(\gamma_H^{1/2} B_H^{-1} \theta)^{1/3} \times W^{1/3} \times \frac{\sigma_r P}{V}$,
- **Bid-Ask Spread**: $k_{TG} = 2\phi(\gamma_H^{1/2} B_H^{-1} \theta)^{-1/3} \times W^{-1/3} \times \sigma_r P$,
- **Average Order Size**: $\frac{\bar{Q}_{TG}}{V} = (\gamma_H^{1/2} B_H^{-1} \theta)^{-2/3} \times W^{-2/3}$.
- **Length of Trading Day**: $H = (\gamma_H B_H \theta^{-1})^{2/3} \times W^{-2/3}$

2. Invariant Bet Frequency - Assumptions

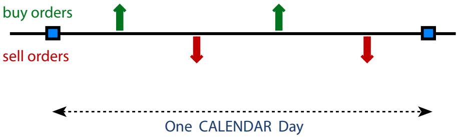
Theory of Invariant Bet Frequency assumes that as **trading activity** varies across stocks,

- **Bet size** B_1 per calendar day **varies** proportionally.
- **Bet frequency** γ_1 per calendar day remains **constant**.

The model implies that all variation in trading activity is explained exclusively by variation in bet size.

2. Invariant Bet Frequency - Intuition

Benchmark Stock with Volume V^*

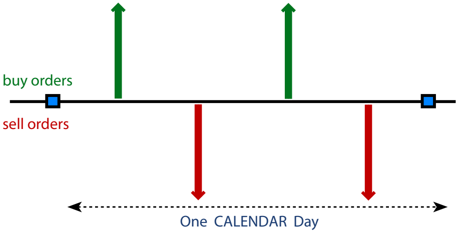


Avg. Order Size $\bar{Q}^*$ as fraction of V^* $= 1/4$

Market Impact of 1/4 V^*

$= 200$ bps $/ \ 4^{1/2} = 100$ bps

Stock with Volume $=$ 8 $\times$ V^*



Avg. Order Size $\bar{Q}$ as fraction of V $= 1/4$

Market Impact of 1/4 V

$= 200$ bps $/ \ 4^{1/2} = 100$ bps

2. Invariant Bet Frequency - Predictions

If **trading activity** increases by **one percent**, then some math implies the following cross-sectional predictions:

- **Average size of a liquidity trade**, as a fraction of average daily volume, is **constant** because order size increases proportionally with average daily volume;
- **Market impact** of trading X percent of average daily volume is **constant**;
- **Bid-ask spread** is **constant**.

2. Invariant Bet Frequency - Comment

We believe that the model of Invariant Bet Frequency is the **"default model"** that implicitly but incorrectly guides the intuition of many asset managers.

- Model justifies trading say no more than 1% of average daily volume for all stocks, regardless of level of trading activity.
- Model justifies imputing same number of basis points in transactions costs for individual stocks in a basket with both active and inactive stocks, where size of trades are proportional to average daily volume.

2. Invariant Bet Frequency - Prediction Math

The Model of Invariant Bet Frequency implies:

- **Market Impact**: $\lambda_\gamma = \theta^{-1}\gamma_1^{1/2} \times W^0 \times \frac{\sigma_r P}{V}$,
- **Bid-Ask Spread**: $k_\gamma = 2\phi\gamma_1^{-1/2} \times W^0 \times \sigma_r P$,
- **Average Order Size**: $\frac{\bar{Q}_\gamma}{V} = \gamma_1^{-1} \times W^0$,
- **Length of Trading Day**: $H = 1 \times W^0$,

where W is the **trading activity**.

3. Invariant Bet Size - Assumptions

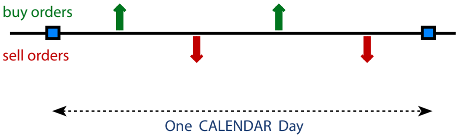
Theory of Invariant Bet Size assumes that as **trading activity** varies across stocks,

- **Bet size** B_1 per calendar day remains **constant**.
- **Bet frequency** γ_1 per calendar day **varies** proportionally.

The model implies that all variation in trading activity is explained exclusively by variation in bet frequency.

3. Invariant Bet Size - Intuition

Benchmark Stock with Volume V^*

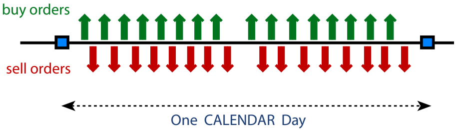


Avg. Order Size $\bar{Q}^*$ as fraction of V^* $= 1/4$

Market Impact of $1/4\ V^*$

$= 200$ bps $/\ 4^{1/2} = 100$ bps

Stock with Volume $=$ 8 $\times\ V^*$



Avg. Order Size $\bar{Q}$ as fraction of V

$= 1/32 = 1/4 \times 8^{-1}$

Market Impact of $1/32\ V$

$= 200$ bps $/\ 32^{1/2}$

Market Impact of $1/4\ V$

$= 8 \times 200$ bps $/\ 32^{1/2}$ bps $= 100$ bps $\times 8^{1/2}$

3. Invariant Bet Size - Predictions

If **trading activity** increases by **one percent**, then some math implies the following cross-sectional predictions:

- **Average size of a liquidity trade**, as a fraction of average daily volume, decreases by **one percent**;
- **Market impact** of trading X percent of average daily volume increase by **1/2 of one percent**;
- **Bid-ask spread** decreases by **1/2 of one percent**.

Intuition: Since there are more independent liquidity trades per day, trading volume increases twice as fast as order imbalances. Thus, market depth increases at half the rate as trading volume.

3. Invariant Bet Size - Prediction Math

The Model of Invariant Bet Size implies:

- **Market Impact**: $\lambda_B = \theta^{-1}(B_1^{-1}\theta)^{1/2} \times W^{1/2} \times \frac{\sigma_r P}{V}$,
- **Bid-Ask Spread**: $k_B = 2\phi(\theta B_1^{-1})^{-1/2} \times W^{-1/2} \times \sigma_r P$,
- **Average Order Size**: $\frac{\bar{Q}_B}{V} = (\theta B_1^{-1})^{-1} \times W^{-1}$,
- **Length of Trading Day**: $H = 1 \times W^0$,

where W is the **trading activity**.

Identification

- Note that the level of market impact, the level of bid-ask spreads, and the average size of liquidity trades are not identified by the theory, but can be estimated from data.

- Note that the length of the trading day H is not identified as well, but we cannot estimate it from data using our methodology either.

Testing - Portfolio Transition Data

The empirical implications of the three proposed models are tested using a proprietary dataset of **portfolio transitions**.

- Portfolio transition occurs when an old (legacy) portfolio is replaced with a new (target) portfolio during replacement of fund management or changes in asset allocation.
- Our data includes 2,680+ portfolio transitions executed by a large vendor of portfolio transition services over the period from 2001 to 2005.
- Dataset reports executions of 400,000+ orders with average size of about 4% of ADV.

Portfolio Transitions and Liquidity Trades

Portfolio transition trades are like **noise (liquidity) trades**.

- Institutional asset managers correspond to liquidity traders most of the time;
- Portfolio transitions are transactions in the differences between portfolios of two different professional asset managers;
- These orders, especially orders resulting from selling off stocks from old portfolios, do not have much hot information.

We use the data on **transition orders** to examine which model makes the most reasonable assumptions about how the frequency of liquidity trades and their size varies with **trading activity**.

Tests for Liquidity Orders Size - Design

All three models are nested into one specification that relates **trading activity** W and the average size of liquidity trade $\bar{Q}$, proxied by **a transition order of** X **shares**, as a fraction of average daily volume V:

$$ln\left[\frac{X_i}{V_i}\right] = \bar{q} + a_0 \times ln\left[\frac{W_i}{W_*}\right] + \tilde{\epsilon}$$

The variables are scaled so that $e^{\bar{q}} \times 10^4$ is the average size of liquidity trade as a fraction of daily volume (in bps) for **a benchmark stock** with:

- daily standard deviation of 2%,
- price of \$40 per share,
- trading volume of 1 million shares per day,
- trading activity $W_* = 2\% \times \$40 \times$ 1 million.

Tests for Liquidity Orders Size - Design

Three models differ only in their predictions about **parameter a_0**.

- **Model of Trading Game Invariance:** $a_0 = -2/3$.
- Model of Invariant Bet Frequency: $a_0 = 0$.
- Model of Invariant Bet Size: $a_0 = -1$.

We estimate the parameter a_0 to examine which of three models make the most reasonable assumptions.

Tests for Liquidity Orders Size - Results

		NYSE		NASDAQ	
	All	Buy	Sell	Buy	Sell
$\bar{q}$	**-5.67*****	-5.68***	-5.63***	-5.75***	-5.65***
	(-342.14)	(-253.47)	(-313.86)	(-174.49)	(-182.49)
a_0	**-0.63*****	**-0.63*****	**-0.60*****	**-0.71*****	**-0.61*****
	(-75.27)	(-61.16)	(-75.28)	(-37.95)	(-49.27)

- **Model of Trading Game Invariance:** $a_0 = -2/3$.
- Model of Invariant Bet Frequency: $a_0 = 0$.
- Model of Invariant Bet Size: $a_0 = -1$.

***is 1%-significance, **is 5%-significance, *is 10%-significance.

Tests for Liquidity Orders Size - F-Tests

		NYSE		NASDAQ	
	All	Buy	Sell	Buy	Sell
Model of Trading Game Invariance: $a_0 = -2/3$					
F-test	17.03	13.74	72.00	6.53	18.56
p-val	0.0000	0.0002	0.0000	0.0107	0.0000
Model of Invariant Bet Frequency: $a_0 = 0$					
F-test	5664.91	3740.45	5667.60	1440.32	2427.51
p-val	0.0000	0.0000	0.0000	0.0000	0.0000
Model of Invariant Bet Size: $a_0 = -1$					
F-test	1920.52	1306.11	2537.08	229.30	966.99
p-val	0.0000	0.0000	0.0000	0.0000	0.0000

Tests for Liquidity Orders Size - Summary

Model of Trading Game Invariance assumes: An increase of one percent in trading activity W leads to a decrease of 2/3 of one percent in size of liquidity trade as a fraction of daily volume (for constant returns volatility).

Results: The estimates provide strong support for Model of Trading Game Invariance. The coefficient predicted to be -2/3 is estimated to be -0.63.

Discussion:

- The assumptions made in our model match the data economically.
- F-test, however, rejects our model statistically because of small standard errors of estimates.
- Alternative models are rejected soundly with very large F-values.

Order Sizes Across Volume Groups

Do the data match models' assumptions across **10 volume groups**?

$$\ln\left[\frac{X_i}{V_{1,i}}\right] = \left[\sum_{j=1}^{10} \mathbb{I}_{j,i} q_j\right] + a_0 \times \ln\left[\frac{W_i}{W_*}\right] + \tilde{\epsilon}$$

- **Parameter** a_0 is restricted to values predicted by each model ($a_0 = -2/3, a_0 = 0$, or $a_0 = -1$).
- Indicator variable $\mathbb{I}_{j,i}$ is one if ith order is in the jth volume groups.
- **Dummy variables** $\bar{q}_j, j = 1, ..10$ quantify the average trade size for a benchmark stock based on data for jth volume group. If assumptions of the model are reasonable, then all dummy variables should be constant across volume groups.

Average Order Sizes Across Volume Groups

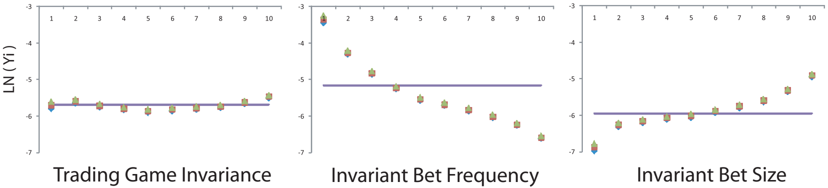


Figure plots **average order size** $\bar{q}_j$ across **10 volume groups**. Group 1 contains stocks with the lowest volume. Group 10 contains stocks with the highest volume. The volume thresholds are 30th, 50th, 60th, 70th, 75th, 80th, 85th, 90th, and 95th percentiles for NYSE stocks.

Tests for Liquidity Orders Size - Summary

Predictions: If the data match assumptions well, then **all dummy variables** $\bar{q}_j, j = 1, .10$ should be constant across volume groups.

Results: The data match the assumptions of Model of Trading Game Invariance much better than the two alternative models.

Discussion:

- Pattern of dummy variables of Model of Trading Game Invariance is reasonably constant.
- But note that in Model of Trading Game Invariance, trade size for largest 5% of stocks is statistically larger than predicted by the model, due to low standard errors.
- Alternative models fail miserably to explain the data on trade sizes.

Tests for Liquidity Orders Size - Conclusion

Data on the **average size** of portfolio transition orders strongly support **assumptions** made in Model of Trading Game Invariance. The data soundly reject assumptions made in alternative models.

Intuition: when trading activity increases, both frequency and size of liquidity trades increase; neither remains constant.

Portfolio Transitions and Trading Costs

We use data on the **implementation shortfall** of portfolio transition trades to test the **prediction** of the three proposed models concerning transaction costs, both market impact and bid-ask spread.

Portfolio Transitions and Trading Costs

"Implementation shortfall" is the difference between actual trading prices (average execution prices) and hypothetical prices resulting from "paper trading" (price at previous close).

There are **several problems** usually associated with using implementation shortfall to estimate transactions costs. Portfolio transition orders avoid most of these problems.

Problem I with Implementation Shortfall

Implementation shortfall is a **biased estimate** of transaction costs when it is based on price changes and executed quantities, because these quantities themselves are often correlated with price changes in a manner which biases transactions costs estimates.

Example: Orders are often canceled when price runs away. Since these non-executed, high-cost orders are left out of the sample, we would underestimate transaction costs.

For **portfolio transitions**, this problem does not occur: Orders are not canceled. The timing of transitions is somewhat exogenous.

Problems II with Implementation Shortfall

The second problem is **statistical power**.

Example: Suppose that 1% ADV has a transactions cost of 20 bps, but the stock has a volatility of 200 bps. Order adds only 1% to the variance of returns. A properly specified regression will have an R squared of 1% only!

For the large database of **portfolio transitions**, this problem does not occur: Large and numerous orders improve statistical precision.

Tests For Market Impact and Spread - Design

All three models are nested into one specification that relates **trading activity** W and **implementation shortfall C** for a transition order for X shares:

$$C_i \times \left[\frac{0.02}{\sigma_r}\right] = \frac{1}{2}\bar{\lambda}\left[\frac{W_i}{W_*}\right]^{\alpha_0}\frac{X_i}{(0.01)V_i} + \frac{1}{2}\bar{k}\frac{(X_{omt,i} + X_{ec,i})}{X_i}\left[\frac{W_i}{W_*}\right]^{\alpha_1} + \tilde{\epsilon}$$

The variables are scaled so that parameters $\bar{\lambda}$ and $\bar{k}$ measure in basis point the market impact (for 1% of daily volume V) and spread for a **benchmark stock** with volatility 2% per day, price $40 per share, and daily volume of 1 million shares.

- Spread is assumed to be paid only on shares executed externally in open markets and external crossing networks, not on internal crosses.
- Implementation shortfall is adjusted for differences in volatility.

Tests For Market Impact and Spread - Design

The three models make different predictions about **parameters a_0 and a_1**.

- **Model of Trading Game Invariance:** $\alpha_0 = 1/3, \alpha_1 = -1/3$.
- Model of Invariant Bet Frequency: $\alpha_0 = 0, \alpha_1 = 0$.
- Model of Invariant Bet Size: $\alpha_0 = 1/2, \alpha_1 = -1/2$.

We estimate a_0 and a_1 to test which of three models make the most reasonable predictions.

Tests For Market Impact and Spread - Results

		NYSE		NASDAQ	
	All	Buy	Sell	Buy	Sell
$1/2\bar{\lambda}$	**2.85***	2.50***	2.33***	4.2***	2.99***
	(11.60)	(4.85)	(6.37)	(5.58)	(4.51)
α_0	**0.33***	**0.18***	**0.33***	**0.33***	**0.35***
	(13.37)	(4.05)	(6.02)	(6.18)	(7.83)
$1/2\bar{k}$	**6.31***	14.99***	2.82*	8.38*	3.94**
	(5.58)	(5.92)	(2.02)	(2.52)	(2.63)
α_1	**-0.39***	**-0.19***	**-0.46***	**-0.36***	**-0.45***
	(-15.73)	(-4.33)	(-7.56)	(-5.85)	(-9.62)

- Model of Trading Game Invariance: $\alpha_0 = 1/3, \alpha_1 = -1/3$.
- Model of Invariant Bet Frequency: $\alpha_0 = 0, \alpha_1 = 0$.
- Model of Invariant Bet Size: $\alpha_0 = 1/2, \alpha_1 = -1/2$.

***is 1%-significance, **is 5%-significance, *is 10%-significance.

Tests For Market Impact and Spread - F-Tests

	All	NYSE Buy	NYSE Sell	NASDAQ Buy	NASDAQ Sell
Model of Trading Game Invariance: $\alpha_0 = 1/3, \alpha_1 = -1/3$					
F-test	2.60	8.57	2.25	0.09	3.12
p-val	0.0742	0.0002	0.1057	0.9114	0.0443
Model of Invariant Bet Frequency: $\alpha_0 = 0, \alpha_1 = 0$					
F-test	176.14	14.77	47.03	33.11	71.06
p-val	0.0000	0.0000	0.0000	0.0000	0.0000
Model of Invariant Bet Size: $\alpha_0 = 1/2, \alpha_1 = -1/2$					
F-test	30.34	39.81	5.23	7.21	5.92
p-val	0.0000	0.0000	0.0054	0.0007	0.0027

Tests for Impact and Spread - Summary

Model of Trading Game Invariance predicts: The coefficient for market impact is $\alpha_0 = 1/3$. The coefficient for bid-ask spread is $\alpha_1 = -1/3$.

Results: Coefficient α_0 is estimated to be 0.33, matching prediction of Model of Trading Game Invariance exactly. Coefficient α_1 is estimated to be -0.39, matching prediction of the model reasonably closely.

Discussion:

- Model of Trading Game Invariance is statistically rejected due to small standard errors and imperfect match for spread.
- Alternative models are soundly rejected.
- For benchmark stock, half-spread is 7.90 basis points and half market impact is 2.89 basis points (restricting α_0 to be $1/3$ and α_1 to be -$1/3$).

Transactions Costs Across Volume Groups

Do the data match models' assumptions across **10 volume groups**?

$$C_i \times \left[\frac{0.02}{\sigma_r}\right] = \left(\sum_{j=1}^{10} \mathbb{I}_{j,i} \times {}^1\!/\!_2 \lambda_j\right) \left[\frac{W_i}{W_*}\right]^{\alpha_0} \frac{X_i}{(0.01)V_i} + \left(\sum_{j=1}^{10} \mathbb{I}_{j,i} \times {}^1\!/\!_2 k_j\right) \frac{(X_{omt,i} + X_{ec,i})}{X_i} \left[\frac{W_i}{W_*}\right]^{\alpha_1} + \tilde{\epsilon}$$

- **Parameter α_0 and α_1** are restricted to values predicted by each model ($\alpha_0 = 1/3, \alpha_0 = -1/3$; $\alpha_0 = 0, \alpha_0 = 0$; or $\alpha_0 = 1/2, \alpha_0 = -1/2$).
- Indicator variable $\mathbb{I}_{j,i}$ is one if ith order is in the jth volume groups.
- **Dummy variables $\bar{\lambda}_j$ and $\bar{k}_j, j = 1, ..10$** quantify the market impact and spread for jth volume group. If assumptions of the model are reasonable, then all dummy variables should be constant across volume groups.

Transactions Costs Across Volume Groups

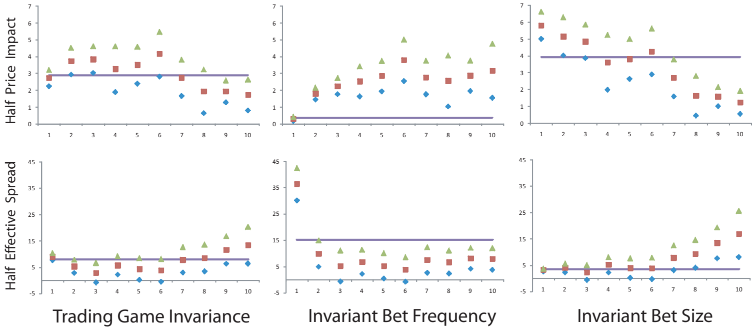


Figure plots **half market impact** $\frac{1}{2}\bar{\lambda}_j$ and **half effective spread** $\frac{1}{2}\bar{k}_j$ across **10 volume groups**. Group 1 contains stocks with the lowest volume. Group 10 contains stocks with the highest volume. The volume thresholds are 30th, 50th, 60th, 70th, 75th, 80th, 85th, 90th, and 95th percentiles for NYSE stocks.

Tests for Impact and Spread - Summary

Predictions: If the data match predictions well, then **all dummy variable** $\bar{\lambda}_j$ **and** $\bar{k}_j, j = 1, ..10$ should be constant across volume groups.

Results: Pattern is more stable for our model of Trading Game Invariance than for other two models.

Discussion:

- High precision for small stock anchors models parameters.
- For model of Trading Game Invariance, most active stocks have less impact and higher spreads than predicted, due to basket trades?
- Model of Invariant Bet Frequency gives more weight to orders in small stocks (since these orders are large relative to volume) and incorrectly extrapolates the estimates for small stocks to large ones. This model does reasonably when small stocks are excluded from the sample.

Conclusions

Our tests provide strong support for the model of Trading Game Invariance which implies, for example, that a **one percent increase** in trading activity $W = V \times P \times \sigma_r$ is associated with ...

- an increase of **1/3 of one percent** in **average order size**,
- an increase of **2/3 of one percent** in its **arrival frequency**,

and leads to...

- an increase of **1/3 of one percent** in **market impact**,
- a decrease of **1/3 of one percent** in **bid-ask spread**.

Practical Implications

For a **benchmark stock**, half market impact $\frac{1}{2}\lambda^*$ is 2.89 basis points and half-spread $\frac{1}{2}k^*$ is 7.90 basis points.

The Model of Trading Game Invariance extrapolates these estimates and allows us to calculate **expected trading costs** for any order of X shares for **any security** using a simple formula:

$$C(X) = \frac{1}{2}\lambda^* \left(\frac{W}{(40)(10^6)(0.02)}\right)^{1/3} \frac{\sigma_r}{0.02}\frac{X}{(0.01)V} + \frac{1}{2}k^* \left(\frac{W}{(40)(10^6)(0.02)}\right)^{-1/3} \frac{\sigma_r}{0.02},$$

where trading activity $W = \sigma_r \times P \times V$

- σ_r is the expected daily volatility,
- V is the expected daily trading volume in shares,
- P is the price.

More Practical Implications

- **Trading Rate:** If it is reasonable to restrict trading of the benchmark stock to say 1% of average daily volume, then a smaller percentage would be appropriate for more liquid stocks and a larger percentage would be appropriate for less liquid stocks.
- **Components of Trading Costs:** For orders of a given percentage of average daily volume, say 1%, bid-ask spread is a relatively larger component of transactions costs for less active stocks, and market impact is a relatively larger component of costs for more active stocks.
- **Comparison of Execution Quality:** When comparing execution quality across brokers specializing in stocks of different levels of trading activity, performance metrics should take account of nonlinearities documented in our paper.

1987 Stock Market Crash

Facts about 1987 stock market crash:

- **Trading volume** on October 19 was **$40 billion** ($20 billion futures plus $20 billion stock). Typical volume was lower (say $20 billion) but inflation makes 1987 dollar worth more than 2001-2005 dollar.
- **Volatility** during crash was extremely high, so **2%** expected volatility per day might be reasonable.
- From Wednesday to Tuesday, **portfolio insurers** sold **$14 Billion** ($10 billion futures plus $4 billion stock).
- From Wednesday to Tuesday, **S&P 500 futures** declined from 312 to 185, a decline of **41%** (including bad basis). **Dow** declined from 2500 to 1700, a decline of **32%**.

1987 Stock Market Crash

Our market impact formula implies decline of

$$2 \times 2.89 \times \left(\frac{40 \times 10^9}{40 \times 10^6} \right)^{1/3} \times \frac{0.02}{0.02} \times \frac{14/40}{0.01} = 2023\, bp = 20.23\%$$

Our model suggests **portfolio insurance selling** had market impact of about **20%**, but you can argue about assumptions concerning implementation details.

More Philosophical Implications

Trades and prices are not completely random. There are similar structures, i.e. "trading games", in the trading data. Trading games are invariant across stocks and across time, except they are played at different pace.