Managing the Risk of Variable Annuities: a Decomposition Methodology

By

Thomas S. Y. Ho Ph.D.

President

Thomas Ho Company Ltd

55 Liberty Street, 4 B

New York   NY 10005

tom.ho@thomasho.com

And

Blessing Mudavanhu Ph.D.

Vice President

Credit Risk Analytics

Merrill Lynch & Company

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Abstract

The market of variable annuities has grown tremendously in recent years and has become a significant part of our capital markets. These equity and interest rate structured products offer a broad range of guarantees, whose risks are typically borne by the insurers' balance sheets. The limited risk capital of the life insurance industry may constrain the future growth of the market, and therefore the management of the risk of these guarantees is an urgent problem to address. In this paper, we apply a decomposition methodology to identify the risks of these guarantees. We then discuss the hedging strategies in managing them within the context of an investment process. Finally, we discuss the broad applications of the methodology.

Managing the Risk of Variable Annuities: a Decomposition Methodology

A. Introduction

Variable annuities are retirement products sold by insurance companies to individuals, qualified and non-qualified accounts. The insurance companies manage the retirement contributions over a duration, the accumulation period, for a fee in their separate accounts. Each policyholder at the end of the accumulation period can choose to receive the account value or a specified annuity.

This basic idea of the variable annuities is relatively simple. But in practice, variable annuities are complex structured equity and interest rate products. The complexity arises from the multitudes of options that insurers offer their policyholders, particularly in the form of guarantees embedded, and not detachable, in these products. Perhaps, the simplest guarantee is called the Guaranteed Minimum Death Benefits (GMDB). In the event that the policyholder dies during the accumulation period, GMDB offers the beneficiary of the policyholder an option to receive the account value or the paid-in premiums plus a guaranteed rate of return. The Guaranteed Minimum Account Value (GMAV) is similar to the GMDB except that the guarantee is not a death benefit. The feature simply offers a guaranteed minimum return of the premium. The Guaranteed Minimum Withdrawal Benefits (GMWB) extends the GMAV in guaranteeing a minimum return on the premiums. It offers the policyholders an option to withdraw a fixed percent of the account at a preset value at each anniversary date over a period of time, providing an early exercise option, as in an American stock option, to the policyholders. The Guaranteed Minimum Income Benefits (GMIB) offers the policyholder the option of receiving the account value or an annuity at the end of the accumulation period. This option leads to a complex mix of equity and interest rate risks embedded in the guarantee. In practice, variable annuities often have combinations of these guarantees with many variations to the basic design, and they have become the key features in filling the needs of the market place.

These guarantees are prevalent in all variable annuities because they provide risk management for individuals bundled in their savings for retirement. Today, the US households hold savings and retirement investments but insure only $1 trillion in managing mortality risk. There is a latent demand for retirement investment that can manage the retirees' life cycle consumption risk: out-living their savings, precipitous falls in asset values, reduction of purchasing power by inflation and more. Variable annuities will continue to evolve to fill these needs, and the potential for growth of the market for these structured products is enormous.

However, in providing these guarantees, the insurance companies have to bear both the market risks and the insurance risks on their balance sheets. As the inforce business continues to grow and the guarantees become more complex, the insurance industry's risk capital would be constrained. There are a number of possible solutions. For example, insurers can hedge their balance sheet risk, re-insure the severity of the potential loss, securitize the structured products, or co-manage the business with asset managers. Or, other financial institutions, banks and asset management companies, can offer competing products.

But any of these solutions necessarily requires a way to identify the risks in the variable annuities. Therefore a methodology to measure and manage the risk of these guarantees is an important issue. To date, research has focused on treating the guarantees of the variable annuities as financial products, using the standard value sensitivities (the Greeks) to measure and manage their risks. For example, Milevsky and Promislow (2001) describes a hedging strategy for GMDB. These approaches are limited because they do not identify the complex nature of the products. They are also often not practical because the continual dynamic hedging can be expensive and may not be effective. Literature has paid scant attention to the problem of decomposing the structured products to standard capital market instruments, in order to identify and manage their risks. This paper seeks to fill this void.

The purpose of this paper is two-fold. First, we propose a risk management strategy for underwriting variable annuities. We do so by identifying a representative variable annuity guarantee as a portfolio of standard capital market instruments, using the results to study the risks of these guarantees and showing how the results can be integrated to the investment process. Second, we propose a decomposition methodology that can decompose a structured product into standard capital market instruments. This method extends the Canonical Decomposition method for interest rate risk (see Ho-Lee (2004)) to incorporate equity risks.

The paper proceeds as follows. Section B describes a model of a GMIB. We use a GMIB as a representative variable annuity guarantee because its decomposition can be generalized to other guarantees. Section C presents our decomposition methodology. We show that a variable annuity and its guarantees can be approximated as a portfolio of capital market instruments, and we then describe the usefulness of such a representation. Section D presents the results of the model. Specifically, we show that a portfolio of swaptions and equity put options can replicate a GMIB quite well. We also show the relationship between the fee level and the GMIB value. Section E uses the decomposition result to determine the hedging strategies. It also provides simulation results showing the robustness of the hedge. Section F discusses some of the practical considerations of the hedging and how it should be implemented within the context of an investment process. Section G contains the conclusions, suggesting the broad implications of the results.

B. A Model of a Guaranteed Minimum Income Benefit (GMIB)

There are many types of guaranteed minimum income benefits (GMIB) of a variable annuity. In this paper, we describe only the salient features of the guarantee that is relevant for the paper. We assume that the variable annuity is a single premium product. The policyholder pays a premium P initially. The premium is invested in an equity index.

The return of the index is given by a martingale process with an expected return of μ and an instantaneous volatility of σ over a one month period. Let S(n) be the index value at time n, where each time period is one month, then

$$S(n+1) = S(n) \exp(\mu - 0.5\sigma^2 + \sigma Z)$$

where Z is a standardized normal distribution.

The index is assumed to be liquid. And therefore, the account value of the annuity follows the stochastic process of the index specified above during the period bracketed by the two fee payment dates.

The fee of the variable annuity is paid monthly, and it is a constant proportion (f) of the account value at the end of each month. At the end of the accumulation period, T, the policyholder can elect to receive the account value or a zero coupon bond, with maturity T*. We use a zero coupon bond instead of an annuity, equal payments over a period of time, for the clarity of exposition. A zero coupon bond can capture the impact of interest rate risks of the variable annuity. We further assume that the policyholders have no mortality risk, do not lapse or seek partial withdrawal. These simplifying assumptions do not affect the results of the analysis because the mortality risk is not related to market risks. There are charges for lapsation to discourage optimal withdrawal in the financial sense and therefore most policyholders are discouraged to lapse. Finally, we assume that the policyholders maximize their wealth.

Our model seeks to capture the key features of the option embedded in the GMIB which is the equity put option feature with a stochastic strike price.

1. Interest Rate and the Equity Models

Our model assures that the GMIB and the variable annuity can be viewed as standard contingent claims on the market interest rate and equity risks. Specifically, the assumption enables us to show that the variable annuity and GMIB fair values are directly and instantaneously related to the market risks, and therefore, they can be replicated by the market instruments. In particular, their values can be derived by the contingent claim "risk neutral" valuation methodology, in that we value them relative to the prices of the market instruments. Specifically, we assume a perfect capital market and use a discrete time model, the binomial lattice model with monthly step size to value the variable annuity and the GMIB.

We use the two-factor Ho-Lee model (2004) to model the interest rate risk. There are many interest rate models and we choose to use this model for the following reasons. First, the Ho-Lee model is arbitrage-free enabling us to relative value the contingent claims. Second, this model has no negative interest rates. This property is important because an interest rate model with negative interest rates would over state the option value on the annuity. Third, the two-factor model enables us to decouple the equity one period returns, which must equal the short-term rate, from the long rate, which determines the annuity value. Any one-factor model assuming the short rate to be perfectly correlated with the long rate would be problematic for the modeling. Fourth, the Ho-Lee model is not a lognormal model, in that the interest rates do not grow exponentially. Unacceptably high interest rates would lead to unrealistic expected equity returns. A description of the Ho-Lee model is provided in Appendix A for the completeness of the exposition.

Using the standard contingent claim valuation methodology, we can assume that "expected returns" equal to the risk-free rate over each period. For simplicity, we assume that the equity does not pay dividends and for each period, the stock returns have zero correlation with the change in interest rates. Let S(n;i,j) be the index value and r(n;i,j) the continuously compounding one month interest rate, derived from the interest rate model, at time n and state (i, j). Since we are using a two factor interest rate model, each state is specified by two indices i and j. Then the equity value at time n+1 is:

$$S(n+1;i,j) = S(n;i,j) \exp\left(r(n;i,j) - 0.5\,\sigma^2 + \sigma Z(n)\right) \qquad (1)$$

where Z is a standardized normal distribution.

Given the interest rate yield curve and the interest rate volatility surface, the two factor interest rate lattice can be specified. Then for any interest rate path taken from the lattice, equation (1) provides a path of the equity index value. The set of interest rates and the equity value for each path is called a scenario.

2. Valuation of the Variable Annuity and the GMIB

Thus far, we have discussed only the behavior of the market instruments. We have not described the variable annuity nor the GMIB. The valuation of the variable annuity begins with the specification of the account value along a scenario path. Let V(n; i, j) be the account value at time n, the value at the end of the nth period, and at state (i, j). Then, the account value at the end of the period is based on the equity returns on V(n; i, j) net of the fees. Therefore, we have

$$V(n+1;i,j) = V(n;i,j)(1-f) \exp\left(r(n;i,j) - 0.5\,\sigma^2 + \sigma Z(n)\right) \qquad (2)$$

By the specification of the variable annuity product, the initial account value is the premium:

$$V(0) = P$$

And at the termination of the accumulation period, since the policyholder seeks to maximize the value of his/her holdings, the insurer must pay to the policyholder,

$$Y(i,j) = \mathrm{Max}\left(B(T,i,j) - V(T;i,j),\ 0\right) \qquad (3)$$

where B is the fair value of the "annuity" and V is the account value at time T

Note that the policyholders, on the other hand, receives in all states,

$$Z(i, j) = \text{Max} (V(T; i, j), B(T; i, j))$$

And

$$V(T; i, j) = Z(i, j) - Y(i, j), \quad (4)$$

That is, the sum received by the policyholder together the payout of the insurer has to equal the account value at the termination date.

The valuation of the variable annuity and the GMIB proceeds as follows. We randomly generate 5,000 scenarios in the Ho-Lee interest rate lattice. For each scenario, we use equation (2) to generation the price path of the account value. As a result, we can determine the fee for each month generated from the account value along each scenario. The present value of the fees net of the guarantee cost for each path is called the pathwise value. The cashflow is discounted along the corresponding interest rate path, using the short term rate of each one month period (the step size of the lattice). The average of the pathwise values over the 5,000 scenarios is the value of the variable annuity. The average of the pathwise values of the guaranteed amount according to equation (3) is the value of the GMIB.

C. The Decomposition Method

Many financial structured products have embedded options. Mortgage-backed securities and collateralized mortgage obligations, for example, are often described as option embedded bonds with options derived from the prepayment risk. Likewise, as we have discussed, the guarantees of the variable annuities also have embedded options. But what are these options? Can the embedded options be approximately described by some of the benchmark securities in the capital markets, like caps/floors, swaptions, equity European put and call options?

The ability to answer the above questions is important. The reasons are (1) in identifying the embedded options as a portfolio of benchmark securities like caps and floors, the representation can be used as an asset benchmark in the investment process to manage the embedded option risk in a portfolio (see Wallace (2000)); (2) the embedded options can be valued relative to the benchmark securities, which tends to have liquid market and fair market prices; (3) the risk of the embedded options can be better described and understood since the behavior of the benchmark securities is generally appreciated.

We extend the Ho – Chen (1994) decomposition methodology for interest rate contingent claims to incorporate equity risks in this paper.

1. Pathwise Values

We have discussed that the average pathwise values is the fair value of a contingent claim. The pathwise value can be interpreted as the present value of the cashflow along that scenario on a risk adjusted basis (Ho-Lee (2004)).

Suppose the scenarios cover all the possible paths in the one factor lattice. Ho – Chen shows that if two securities have the same pathwise value and have the same cash flow along the forward curve, then they must also have the same cashflow along each path of the lattice. That is, the two securities are identical according to the lattice model.

Intuitively, suppose that two securities have the same pathwise value for a large set of scenarios in the lattice, and further suppose that the lattice approximates the scenarios in the real world. That means, if we reinvest any payments of either securities at the prevailing one period interest rate at each node, the two securities must have the same future value at some distant future. And in this sense, the two securities are equivalent. The decomposition methodology seeks to determine a portfolio of securities which is equivalent to the structured product. This portfolio is constructed from a predetermined set of benchmark securities.

2. Hedging Instruments

To determine the initial set of benchmark securities to construct the replicating portfolio of the GMIB, we begin with investigating the embedded options in the GMIB.

Suppose that there is no interest rate risk. Then the GMIB is an equity put option. The insurer has sold an equity put option to the policyholder with the expiration date at the end of the accumulation period. The strike is the present value of the annuity on the expiration date.

Suppose that the underlying equity has no risk. Then the equity index becomes a cash account, and that the premium is invested in cash. In this case, the GMIB is similar to a bond call option. At the end of the accumulation period, the policyholder has an option to exchange the "annuity" for the account value. When interest rates remain low, the annuity value would be high at the end of the accumulation period and the account value would be low. The policyholder would elect to take the annuity at a cost to the insurer.

To replicate the embedded equity options, we use a series of equity put options on the equity index of different strike prices with the same expiration date, the end of the accumulation period. To replicate the interest rate risk, we use the bond call options with different strike prices but with the same expiration date. These hedging instruments are provided in Appendix B.

Let the index k = 1,…, 5000 denote the scenarios. Equity_Put(X,k) and Bond_Call(X,k) denote the pathwise value of the put option on equity and the call option on the bond with strike price X for scenario k, respectively. Then we seek to determine the optimal portfolio of the hedging instruments such that the portfolio of which can replicate the GMIB, on the pathwise basis. Specifically,

$$GMIB(k) = a + b(1)* Equity\_Put(70,k) +\ldots b(15)*Equity\_Put(185,k) + c(1)* Bond\_Call(35,k) + \ldots + c(10)*Bond\_Call(125,k) + e(k) \quad (5)$$

The pathwise values of the GMIB and the hedging instruments are calculated. We can use a regression to determine the coefficients. The value of the intercept a is the cash value in $, since cash has a constant value under all scenarios, as its interests accrued over each period must equal to its discount rate. The coefficients b and c are the number of the hedging instruments used in the hedging portfolio.

3. Step-wise Regressions

To search for the optimal hedging portfolio, a step-wise regression is used. The stepwise regression is a technique for choosing the variables, i.e., terms, to include in a multiple regression model. In particular, we use the forward stepwise regression, which starts with no model terms. At each step it adds the most statistically significant term (the one with the highest t-statistic or lowest p-value) until there are none left. This iterative process allows us to identify the hedging instruments that can reduce the R squared significantly.

D. Results of the Decomposition of GMIB and the Variable Annuity

The simulation results in this section assume the market parameters on July 31, 2002. The yield curve is USD zero swap. The interest rate volatility surface is estimated from the traded swaptions volatility surface given by the Black volatilities quoted (in %).

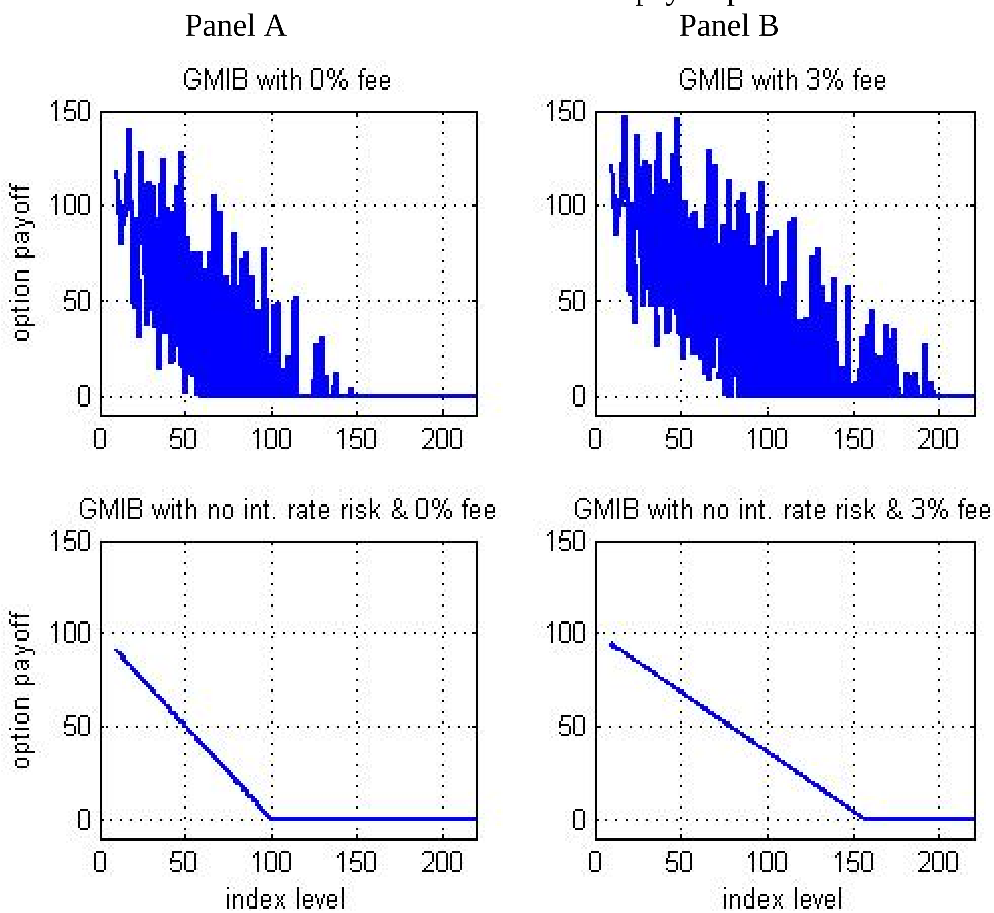
	Swap Tenor (years)									
Option Term (years)	1yr	2yr	3yr	4yr	5yr	6yr	7yr	8yr	9yr	10yr
1yr	44.40	36.50	32.90	29.70	27.60	25.70	25.40	24.00	23.40	23.30
2yr	31.20	28.80	27.10	25.30	24.20	22.90	23.00	21.80	21.30	21.10
3yr	27.00	25.30	24.30	23.10	22.20	21.30	21.10	20.40	20.00	19.90
4yr	24.00	22.80	21.90	21.20	20.70	19.90	19.70	19.10	18.70	18.60
5yr	22.30	21.30	20.80	19.90	19.40	18.70	18.50	18.00	17.70	17.40
7yr	19.80	19.10	18.50	18.10	17.60	16.90	16.70	16.30	16.00	15.80
10yr	17.40	16.40	15.90	15.60	15.10	14.50	14.50	14.10	13.90	13.70

The equity volatility σ is assumed to be 20%.

1. Decomposition of the GMIB

Before we present the results of the decomposition of the GMIB, we first show the payoff profile of a GMIB and a GMIB with no interest rate risk.

Figure1: GMIB and GMIB without interest rate risk payoff profiles.



Panel A of Figure 1 compares the GMIB values at the end of the accumulation period over a range of account value under interest rate risk with that under no interest rate risks. The results clearly show that without interest rate risks, the GMIB is a standard put option. The stochastic interest rates lead to a stochastic strike price. Panel B repeats the analysis of Panel A with the introduction of a 3% fee. The result shows that the higher fee results in using a more out-of-the-money equity put option to characterize the GMIB. Given the motivation of the results in Figure 1, we now determine the decomposition of the GMIB. The simulation also shows that the fees raise the GMIB value. The GMIB value without fees is 4.47% of premium compared to 7.78% of GMIB with 3% annual fee.

Proposition 1. Using the decomposition method described above, the decomposition of the GMIB as a portfolio of the hedging instruments is presented in Table 1 and Table 2 below.

Table 1: Decomposition of the GMIB with no interest rate risk and no fees

A	B	C	D	E
Hedging Instrument	Strike	Dollar Value	Percentage Position	t-statistic
Cash		0	0.00	-5.36
Equity Put	100	3.89	100.00	Infinity

The value of the GMIB with no interest rate risk is 3.89% of the premium and the R-square is 100%. This is an expected result since with no interest rate risk the variable annuity guarantees a return of premium (100), which means that the GMIB option is simply a European put option with a strike price of 100.

Table 1: Decomposition of the GMIB with a 3% annual fee

A	B	C	D	E
Hedging Instrument	Strike	Dollar Value	Percentage Position	t-statistic
Cash		2.3	29.56	9.36
Equity Put	80	0.38	4.88	6.06
Equity Put	130	1.96	25.19	4.03
Equity Put	150	1.37	17.61	1.81
Equity Put	185	3.65	46.92	9.14
Bond Call	35	-42.3	-543.70	-16.01
Bond Call	45	37.87	486.76	14.26
Bond Call	75	1.28	16.45	3.06
Bond Call	95	1.03	13.24	6.58
Bond Call	125	0.24	3.08	8.37

The value of the GMIB is 7.78% of the premium and the R-squared is 94.7%

Column A identifies the hedging instruments used in the decomposition. All the equity put options expires in 15 years. The bond call options are options on the 10 year zero coupon bond, expiring in 15 year also. Column B provides all the strike prices of the options. Column C is the dollar amount in each of the hedging instrument, and it is determined as the product of the regression coefficient and the fair value of the hedging instrument. Column D is the percentage of each hedging instrument position to the GMIB value. Column E is the t statistics showing the importance of the hedging instrument in the replication.

The total value of the GMIB is 7.78% of the premium. The R squared of 94.7% suggests that the hedging fits the pathwise values of the GMIB quite well. The decomposition results are quite intuitive. Without interest rate risk, we have suggested that the GMIB is a basic GMAV and equivalent to an equity put option as shown by Figure 1. However, given the interest rate risks, the "strike price" of the put option is a bond and must necessarily stochastic. This is reflected by the use of put options with strike prices 80,130,150 and 185. Also, we have discussed that GMIB is exposed to the interest rate risk, particularly when the interest rates are low. This is captured by the in-the-money bond call option with strike 35, 45, 75, 95 and out-of-the-money bond call option with a strike price of 125. Finally, since the account value accrues at the stochastic short-term rate, the GMIB value is path dependent. The account value at a node on the expiration date is dependent on the interest rate paths that it has taken. For that reason, the replicating portfolio has a significant cash position, with the short position in the equity put option.

To provide better insights into the effectiveness of the decomposition, Figure 2 depicts the scattered plots of the GMIB pathwise values against those of the replicating portfolio. The results show that the residuals are not proportional to the size of the pathwise values and therefore the replication is more effective for the "worse scenarios" where the insurers have to pay more benefits.

Figure 2: Scatter plot of fitted GMIB pathwise values against GMIB pathwise values.

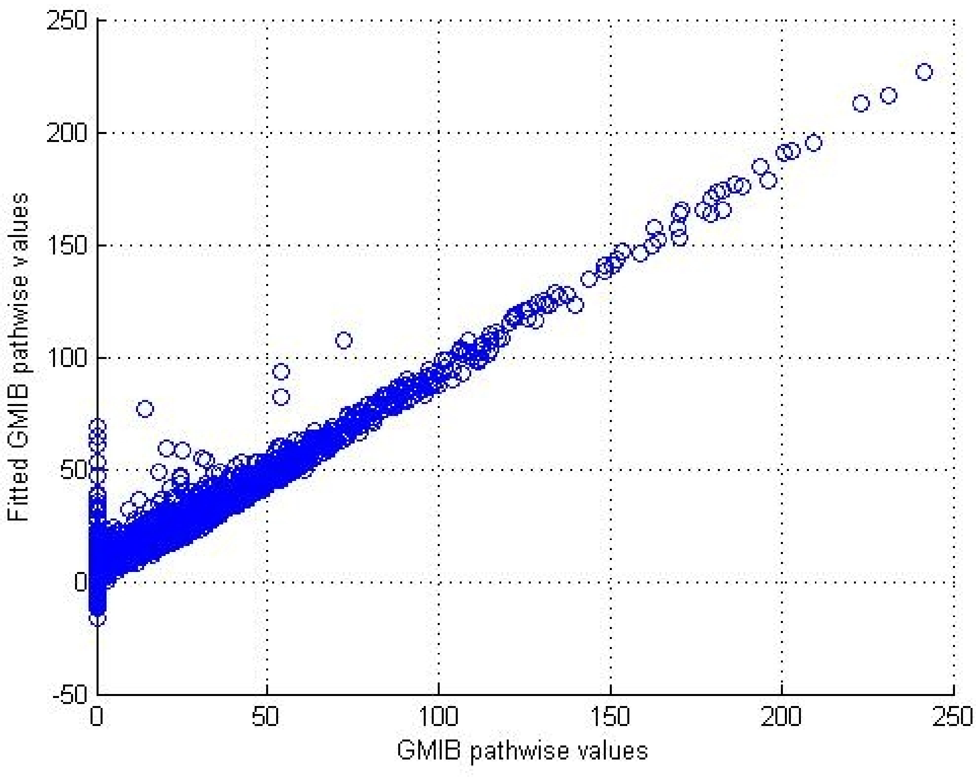


Table 2 below shows that the GMIB option delta and duration are well matching. The results below are calculated using results in appendix B. The replication portfolio delta and duration are just a linear combination of deltas and durations of instruments in the basket replicating portfolios.

Table 3: Delta and Duration Matching

	Delta	Duration
GMIB option	-0.0989	90.9
Replicating Portfolio	-0.0976	97.3

The results show that the replicating portfolio has similar delta and duration to those of the GMIB.

2. Analysis of the Fees

The higher the fees the more valuable the GMIB option. In the presence of fees the GMIB option payoff is given by

$$\max(B(T;i,j)-V(T;i,j),0)=(1-f)^{T-1}\max(B^{*}(T;i,j)-S(T;i,j),0)$$

where $B^{*}(T;i,j)=\dfrac{B(T;i,j)}{(1-f)^{T-1}}$. The above result shows that fees proportionally increase the bond par value thereby increasing the money-ness of the GMIB option.

Proposition 2. Closed-Form Solution for Present Value of Fees

Let F be the present value of GMIB fees, which is a fixed portion (f) of account value. Let the initial premium be P. Then in an arbitrage-free movement of both interest rates and equity index,

$F=(1-(1-f)^{n})P$ where n is the number of years till the termination date or the accumulation period.

Proof:

Consider a particular scenario k. Let the vector of the stock index values at the end of each month along this scenario from time 0 till the end of the accumulation period T be (S(0, k), S(1, k), S(2, k) … S(T, k)). Then the corresponding vector of account value is

$(S(1,k), S(2,k)(1-f), S(3,k)(1-f)^{2}, \ldots, S(T,k)(1-f)^{T-1})$ and the fees is

$(S(1,k)f, S(2,k)(1-f)f, S(3,k)(1-f)^{2} f, \ldots, S(T,k)(1-f)^{T-1}f)$

Let the vector of discount factors for the cashflows for this scenario (k) is

(d(1, k), …… , d(T, k))

Then the pathwise value of the fees for this scenario (k) is

PWV(k) = d(1, k)S(1, k)f+ d(2, k)S(2, k)(1-f)f+ d(3, k)S(3, k)$(1-f)^2$ f+…+d(T, k)S(T, k)$(1-f)^{T-1}$f

But for each i= 0,…, T

$$S(0)=\frac{1}{N}\sum_{k=1}^{N}d(i,k)S(i,k) \text{ for each i.}$$

Hence,

F = S(0) (f (1+ (1-f) + $(1-f)^2$ +…+$(1-f)^{T-1}$))

Noting that S(0) = P. In simplifying, we get the desired result. QED

Corollary: Decomposition of the Variable Annuity

A variable annuity is a portfolio of portion of the equity index and the GMIB.

Proof

The value of the variable annuity is the present value of the inflow of the fees net of the GMIB. Therefore, the variable annuity can be represented by

VA(f) = F(f) – GMIB(f)

Using proposition 2 and the decomposition results, we can now represent the variable annuity as a portion of the equity index and a portfolio of equity and bond options.

VA(f) = g(f) S – GMIB(f)

QED.

The present value of the fees is always a portion (g) of the premium. For example, if the fee is 3%. Then the portion (g) is 36.6%. The present value of the fees (or revenues) is positively related to the fee level. The GMIB is also positively related with the fee level, with a convexity. Their values for different fee (f) levels are presented below.

Table 4: Variable Annuity Values

Fees	0%	1%	2%	3%
PV of fees	0.00	5.83	19.06	30.54
GMIB	4.47	5.47	6.57	7.78
VA	-4.47	0.36	12.49	22.76

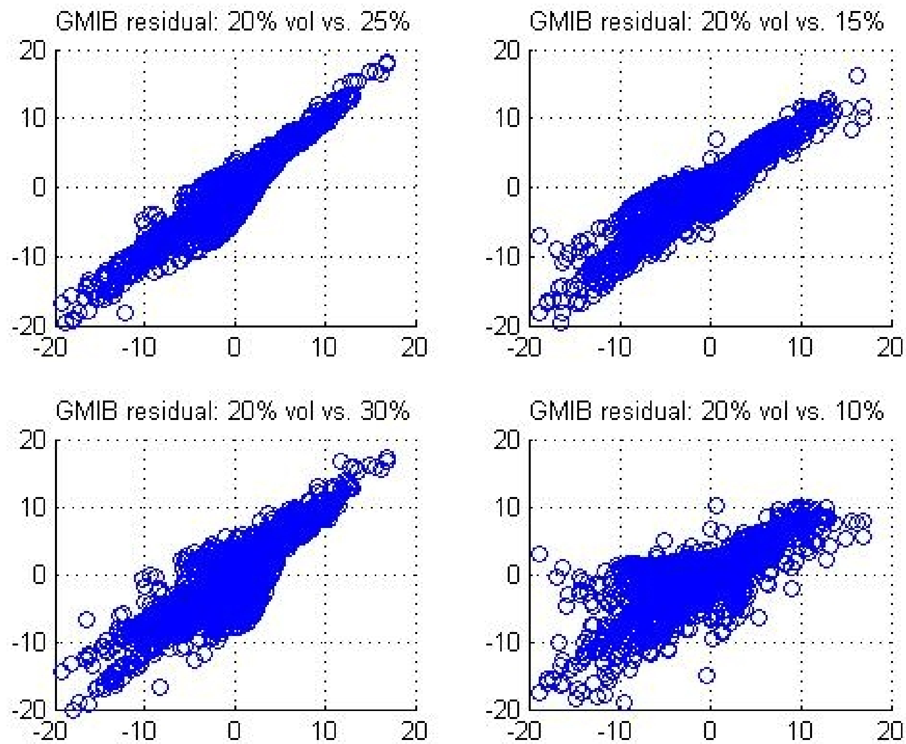
The result shows that the GMIB value increases with the fee, dampening the increase value of the variable annuity to the insurers.

E. Effectiveness of the Static Hedge

1. Sensitivity to the volatility

The idea of the decomposition is to replicate the GMIB with standard hedging instruments. Unlike the dynamic hedging, this hedging approach should offer a more stable hedging strategy. We now show that this hedging is relatively stable by simulations. Figure 3 below shows that the stability of the hedging based on the portfolio constructed using a 20% equity volatility under different market volatilities. A linear trend shows that the optimal hedging portfolio is robust under different volatilities.

Figure 3: Hedge effectiveness.



2. Hedging the Downside Risk – Re-insurance of the Guarantees

We have shown how one can determine the decomposition of the GMIB. However, insurers may not want to hedge the entire GMIB exposure. They may only seek to hedge the tail end of the scenarios, which result in largest losses.

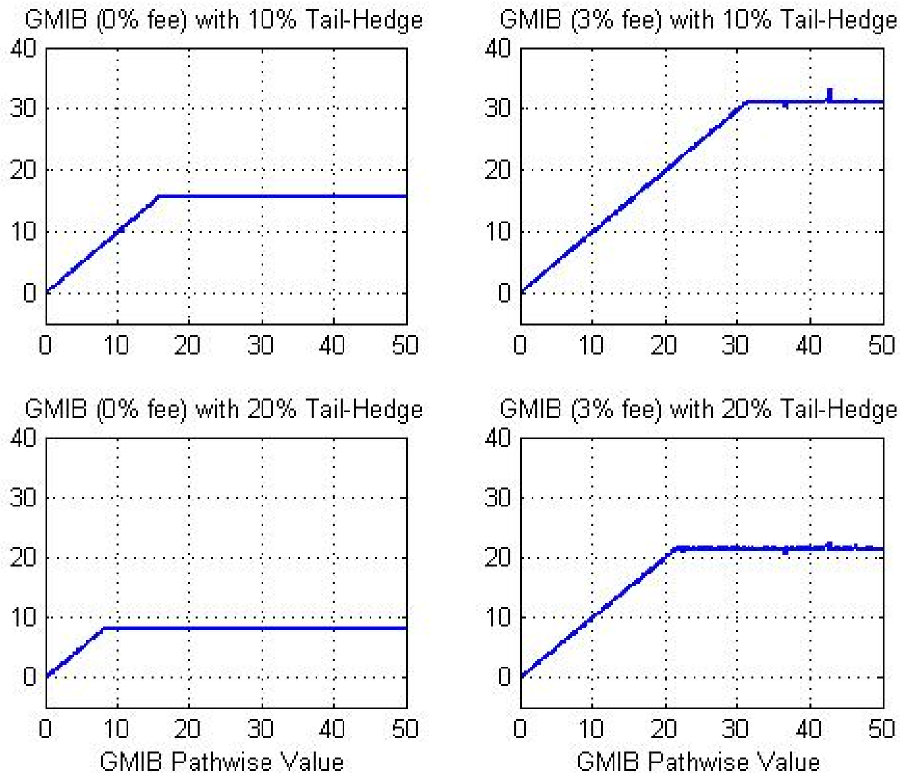
That is, we can order the pathwise values of GMIB, and seek to find the hedging portfolio that would replicate the one sided tail of the GMIB pathwise values. In hedging the 10% tail distribution, we have the following results.

Table 5: Hedging downside risk (10% tail)

A	B	C	D	E
Hedging Instrument	Strike	Dollar Value	Percentage Position	t-statistic
Cash		1.80	49.72	8.8
Equity Put	110	1.38	38.12	13.14
Equity Put	130	0.70	19.34	25.7
Bond Call	100	1.22	33.70	13.52
Bond Call	70	3.79	104.70	20.31
Bond Call	55	-5.27	-145.58	-15.67

The value of the hedging instruments used here is 46% of GMIB value. The R-square is 99.85%. Figure 4 below shows the effectiveness of hedging downside risk for 10% tail and 20% tail hedging with different fee structures. The results show that the hedging is more effective in hedging the tail of the distribution of the GMIB.

Figure 4: Hedge effectiveness.



3. Extending the Method to Other Guarantees

In the introduction, we have discussed that there are many types of guarantees in the variable annuities. The purpose of the paper is to show how the decomposition method enables us to replicate the guarantees or the variable annuities as a portfolio of standard hedging instruments. It is not the purpose of the paper to discuss the decomposition of different types of variable annuities. However, to show the applicability of the decomposition method to variable annuities in general, we describe briefly how some of the standard guarantees can be approximated as standard hedging instrument.

The Guaranteed Minimum Death Benefit (GMDB) in essence offers the policyholder an “option” to receive the maximum of the account value and death benefit amount. GMDB is different to the standard option. For the standard option, the option holder has the right to exercise the option on the expiration date. By way of contrast, the policyholder can “exercise” the option when he dies. From the insurer’s perspective, given a sufficiently large pool of policyholders, the mortality risk is relatively small. Therefore, the insurers have sold a portfolio of equity put options in providing the GMDB. The decomposition method enables us to determine the optimal mix of equity put options in replicating the guaranteed risks.

The Guaranteed Minimum Withdrawal Benefit (GMWB) in essence offers a guarantee on the withdrawal amount. While the actual products have more complicated features, GMWB offers the policyholders to withdraw a fixed amount at each of a selection of anniversary dates, at the maximum of a fixed value and the account value. Therefore, this product in essence offers the policyholders a series of equity put options.

The Guaranteed Minimum Account Value (GMAV) is the GMIB that we have discussed without the interest rate risks.

F. Practical Considerations of the Hedging Strategy

Thus far, we have concentrated on describing the decomposition method applying the variable annuities, the guarantees of the annuities and the downside risks of the guarantees. We have discussed how such a decomposition method can be used for hedging the risks of the products.

The decomposition method is a useful tool in identifying the risks of the product in terms of the standard hedging instruments. In applying the tool in practice, we have to take the investment process into consideration. The decomposition method should be employed within the context of the investment process. To illustrate, let us consider one investment process, where the process begins with management deciding on the investment goals. Then the portfolio strategies are developed where the benchmarks are provided to the portfolio managers. Finally the portfolio performance is monitored. We now discuss how the decomposition method is used within such an investment process.

1. Determining the Investment Objective: Risk and Return Analysis

The method that we have described enables us to determine the market value of the structured product, whether it is the variable annuity, the guarantee or the downside risk of the guarantee. Such a decision must take the profitability of the product, the risk tolerance of the management and the cost of hedging into account. The valuation model enables the management to decide for example whether to eliminate the guaranteed risks, or, re-insurance part of the risks. The actionable decision would be to determine the level of hedging required.

2. Benchmark Construction

Given the risk tolerance determined by the senior management, portfolio strategy group would determine the benchmark portfolio (or the liability benchmark). The decomposition method is then used to determine asset benchmark for the portfolio managers.

3. Hedging Portfolio Revision

The constructed benchmark cannot be used for hedging. This is because the benchmark portfolio may not be able to dynamically hedge the GMIB effectively. We need to determine an appropriate hedging portfolio that matches the comparative statistics of the benchmark portfolio by adjusting the benchmark portfolio. The decomposition does provide the benchmark to formulate the actual hedging strategy.

4. Feedback Control

Return attribution should be used over regular intervals to ensure that the investment process is well designed. Using the tracking errors of the portfolio returns based on the hedging strategies against the portfolio benchmark, we can detect the model risks. Portfolio management performance is monitored taking the transaction costs involved in the hedging into account.

G. Conclusions

This paper contributes to the literature in three main areas. First, using a contingent claim valuation approach, we have analyzed the embedded option risk of the variable annuities, GMIB in particular. We have shown that the embedded option is an equity put option with a stochastic strike price, which is interest rate sensitive. We have shown that the variable annuity is therefore a portfolio of equity, equity put options, and bond options. The level of fee can significantly affect the GMIB value.

Second, we have shown that the decomposition method can determine the hedging portfolio of the tail risk of the GMIB. This approach enables the insurer to “re-insure” the GMIB risks on its balance sheet. Further, the analytical results can be used within the context of the investment process in determining the asset benchmark in portfolio management and measuring the “defects” in the return attribution process, in the feedback loop.

Third, we have extended the Ho and Chen (1994) arbitrage-free bond canonical decomposition to hedge against combined stock and interest rate risks. We have shown that the results are quite robust in hedging and are consistent with intuitions. This decomposition method should find broad applications beyond managing the risks of the annuities products. The method offers an effective way to identify the option risk embedded in structured products in terms of the standard hedging instruments, caps/floors and swaptions. This method also provides a method to value the structured products with embedded options relative to the values of liquid derivatives.

This paper also suggests broader applications to the variable annuity market. Using the decomposition method, insurers can hedge or partial hedge the risk of their guarantees on their balance sheet. Therefore, they can increase the capacity of their risk capital. Further, the methodology may enable them to design variable annuities to meet the market needs. Finally, the use of this methodology does not have to be confined to the insurers. Asset managers can also sell products similar to the variable annuities to enable individuals to manage the risk of their life cycle consumption needs.

Appendix A: The Two Factor Generalized Ho Lee Model

Let $P^n_{i,j}(T)$ be the price of a T year bond at time n, at state (i, j). Then the bond price is specified by combining two one-factor models. Specifically, we have

$$P^n_{i,j}(T) = \frac{P\ (n+T)}{P\ (n)} \prod_{k=1}^{n} \frac{(1+\delta^{k-1}_{0,1}(n-k))}{(1+\delta^{k-1}_{0,1}(n-k+T))} \frac{(1+\delta^{k-1}_{0,2}(n-k))}{(1+\delta^{k-1}_{0,2}(n-k+T))} \prod_{k=0}^{i-1} \delta^{n-1}_{k,1}(T) \prod_{k=0}^{j-1} \delta^{n-1}_{k,2}(T)$$

(A.1)

where

$$\delta^n_{i,1}(T) = \delta^n_{i,1}\delta^{n+1}_{i,1}(T-1)\left(\frac{1+\delta^{n+1}_{i+1,1}(T-1)}{1+\delta^{n+1}_{i,1}(T-1)}\right),$$

$$\delta^n_{i,2}(T) = \delta^n_{i,2}\delta^{n+1}_{i,2}(T-1)\left(\frac{1+\delta^{n+1}_{i+1,2}(T-1)}{1+\delta^{n+1}_{i,2}(T-1)}\right). \quad (A.2)$$

and the one period forward volatilities are given by

$$\delta^m_{i,1}(1) = \delta^m_{i,1} \text{ by definition of } \delta^m_{i,1},$$

We have,

$$\delta^m_{i,1} = \exp\left(-2\cdot\sigma_1(m)\min\left(R^m_{i,1}, R\right)\Delta t^{3/2}\right).$$

Similarly, we can define $\delta^m_{i,2}$ for the other factor, and we have

$$\delta^m_{i,2}(1) = \delta^m_{i,2} = \exp\left(-2\cdot\sigma_2(m)\min\left(R^m_{i,2}, R\right)\Delta t^{3/2}\right) . \quad (A.3)$$

Using the direct extension, we can specify the one period rates for the two factor model for any future period m and state i, and $R^m_{i,1}$ and $R^m_{i,2}$ are defined by

$$R^m_{i,1}\Delta t = -\log\left(\frac{P(n+1)}{P(n)}\right) + \sum_{k=0}^{n-1}\log\left(\frac{(1+\delta^{k-1}_{0,1}(n-k))}{(1+\delta^{k-1}_{0,1}(n-k+T))}\right) + \sum_{j=0}^{i-1}\delta^{n-1}_{k,1}(T),$$

and

$$R_{i,2}^{m}\Delta t = -\log\left(\frac{P(n+1)}{P(n)}\right) + \sum_{k=0}^{n-1}\log\left(\frac{(1+\delta_{0,2}^{k-1}(n-k))}{(1+\delta_{0,2}^{k-1}(n-k+T))}\right) + \sum_{j=0}^{i-1}\delta_{k,2}^{n-1}(T)\,. \qquad (A.4)$$

Appendix B. The Set of Hedging Instruments and their Comparative Statistics

Hedging Instrument	Strike	PV	Delta	Duration: (1/P)*(dP/dr)	Regression Model Coeffs
Cash		1	0	0	2.3001
Equity Put	70	1.3869	-0.0295	46.7920	0.0000
Equity Put	80	2.0828	-0.0402	43.9286	0.1813
Equity Put	90	2.9202	-0.0517	41.5593	0.0000
Equity Put	100	3.8941	-0.0645	39.7583	0.0000
Equity Put	110	5.0154	-0.0811	39.2438	0.0000
Equity Put	120	6.2975	-0.0989	38.5295	0.0000
Equity Put	130	7.7175	-0.1146	37.2075	0.2543
Equity Put	140	9.2571	-0.1289	35.7780	0.0000
Equity Put	150	10.8957	-0.1429	34.5740	0.1260
Equity Put	160	12.6435	-0.1604	33.9993	0.0000
Equity Put	165	13.5556	-0.1681	33.5295	0.0000
Equity Put	170	14.4943	-0.1773	33.3044	0.0000
Equity Put	175	15.4615	-0.1875	33.1237	0.0000
Equity Put	180	16.4547	-0.1960	32.7478	0.0000
Equity Put	185	17.4756	-0.2068	32.6196	0.2087
Bond Call	35	21.4047	0.0000	31.3554	-1.9764
Bond Call	45	17.4690	0.0000	34.8718	2.1676
Bond Call	55	13.6860	0.0000	39.4787	0.0000
Bond Call	65	10.2559	0.0000	45.0696	0.0000
Bond Call	75	7.3480	0.0000	51.5642	0.1742
Bond Call	85	5.0439	0.0000	58.0438	0.0000
Bond Call	95	3.3381	0.0000	65.1357	0.3091
Bond Call	105	2.1571	0.0000	71.6672	0.0000
Bond Call	115	1.3581	0.0000	79.2824	0.0000
Bond Call	125	0.8278	0.0000	87.9010	0.2957
Variable Annuity					
GMIB		7.7780	-0.0968	90.8974	

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