

# Market Equilibrium in a NonCAPM World

Harry Markowitz

The Q-Group  
October 16, 2006

See

“Market Efficiency: A Theoretical  
Distinction and So What?”

Financial Analyst Journal 2005

Vol. 61, No. 5 pp. 17-30

“Financial Market Simulation”

Journal of Portfolio Management

30 th Anniversary Issue

September 2004, pp 142-152

Bruce Jacobs, Ken Levy, and Harry Markowitz

CAPM is an elegant theory.

With the aid of some simplifying assumptions  
it reaches dramatic conclusions about  
practical matters.

For example

How to choose an investment portfolio?

Just buy the market.

How to forecast expected returns?

Just forecast betas.

How to price a new security ?

(see above)

CAPM's simplifying assumptions make it easier to deduce properties of market equilibria.

Like computing trajectories assuming there is no air.

But...

before betting the ranch that the feather and the brick will hit the ground at the same time, it is best to consider the implications of some of the omitted complexities.

## This lecture

- (1) Mostly explores the implications of generalizing *one* of the CAPM's simplifying assumptions.
- (2) Briefly describes a program for computing market equilibria for complex market models.

Distinguish between the statement

“the market is efficient,” in the sense that market participants have accurate information and use it correctly to their benefit,

and the statement

“the market portfolio is a mean-variance efficient portfolio”.

Under some assumptions the two statements are equivalent:

Specifically, if we assume

- A1. Transaction costs and other illiquidities can be ignored
- A2. All investors hold mean-variance efficient portfolios
- A3. All investors hold the same (correct) beliefs about means, variances, and covariances of securities, and--in addition--
- A4. Every investor can lend all she or he has or can borrow all she or he wants at the risk-free rate



Then Conclusion 1 follows:

C1. The market portfolio is a mean-variance efficient portfolio

C1 also follows if A4 is replaced by A4'

A4'. Investors can sell short without limit and use the proceeds of the sale to buy long positions

For example, A4' says that any investor can deposit \$1,000 with a broker, short \$1,000,000 worth of one security, and buy long \$1,001,000 worth of another security.

In addition to C1, A1 through A4 (or A1 through A4') imply

C2. In equilibrium, the expected return for each security depends only on its beta (the regression of its returns against the return on the market).

This relationship between the security's expected return and its beta is a simple linear relationship.

C2 is the basis for the CAPM's prescriptions for risk adjustment and asset valuation.

C1 and C2 do *not* follow from assumptions A1 through A3 if A4 (or A4') is replaced by a more realistic description of the investor's investment constraints.

Example,

This first example assumes that investors cannot sell short or borrow.

Later we note that the same results hold if investors *can* borrow limited amounts or *can* sell short but are subject to Reg T or a similar constraint.

The example assumes A1 through A3.

# Table 1

## Example Inputs

Security	Expected Return	Standard Deviation
1	0.15%	0.18%
2	0.10%	0.12%
3	0.20%	0.30%

The example assumes returns are uncorrelated.

Later we note that similar results follow for correlated risks.

Assume that the investor can choose any portfolio that meets the following constraints:

$$X_1 + X_2 + X_3 = 1.0 \quad (1)$$

and

$$X_1 \geq 0, X_2 \geq 0, X_3 \geq 0 \quad (2)$$

In Figure 1,  $X_1$ -- the fraction invested in Security 1--is plotted on the horizontal axis;  $X_2$  -- is plotted on the vertical axis; and  $X_3$  -- the fraction invested in the third security--is given implicitly by the relationship

$$X_3 = 1 - X_1 - X_2$$

The market portfolio  $M$  is not efficient for either the investor with or without nonnegativity constraints (2)

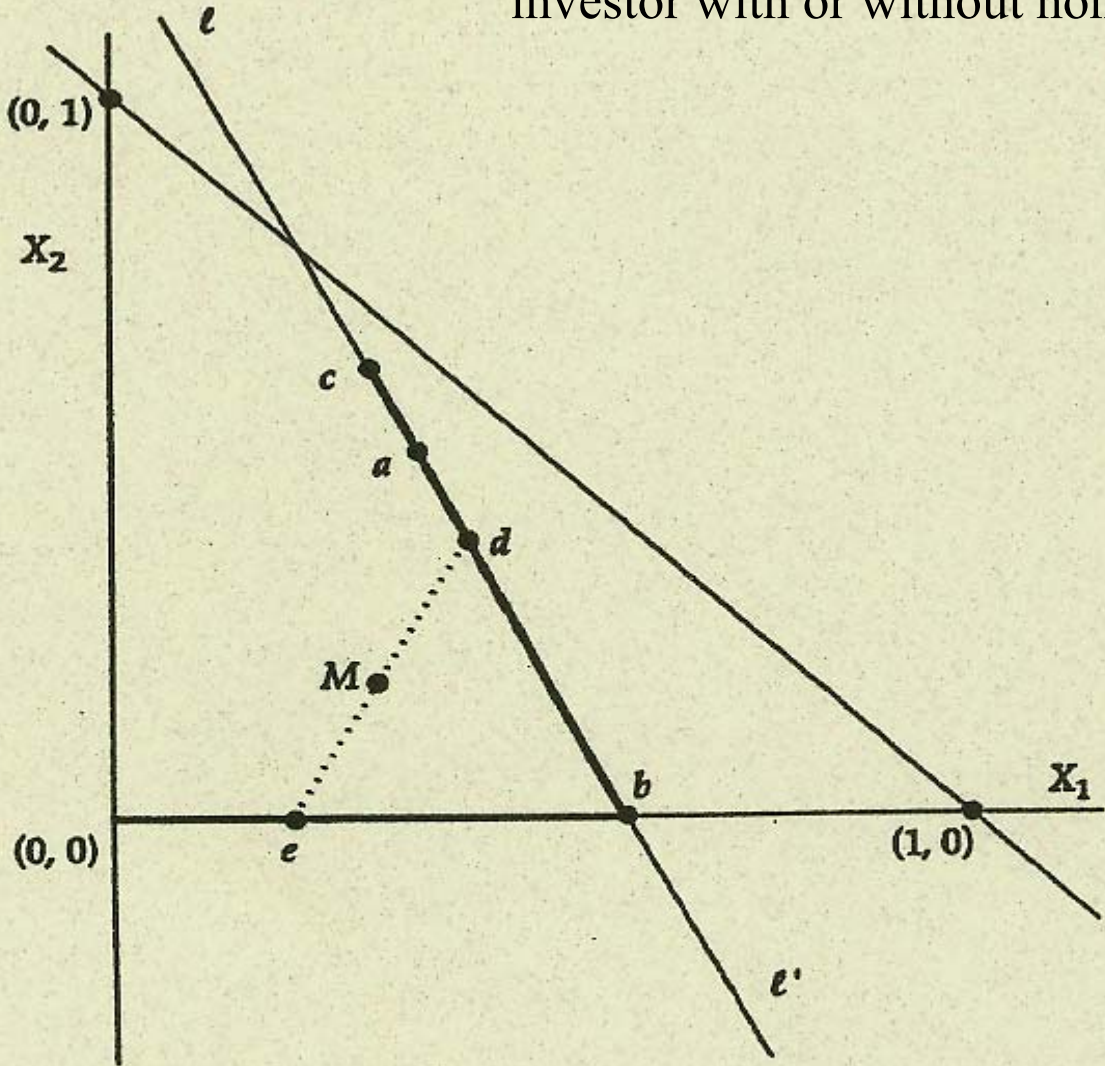


Figure 1 shows that *if* Table 1 were equilibrium beliefs *then* the market portfolio would not be mean-variance efficient portfolio.

*But* can these be equilibrium beliefs?

Consider the following Simple Market:



Inhabitants of an island live on coconuts and the produce of their own gardens.

The island has three enterprises, namely, three coconut farms.

Once a year, a market convenes to trade the shares of the three farms. Each year, the resulting prices of shares turn out to be the same as those of preceding years.

Thus, the only source of uncertainty of return is the dividend each stock pays during the year--which is the stock's pro rata share of the farm's production.

It is shown in Appendix of the paper that means, variances and covariances of coconut production exist that imply the efficient set in Figure 1-- or any other three-security efficient set that we cite.

With such a probability distribution of returns, the market is rational, in the sense that each participant knows the true probability distribution of returns and each seeks and achieves mean-variance efficiency.

Nevertheless, in contrast to the usual CAPM conclusion, the market portfolio is not an efficient portfolio.

It also follows that there is no representative investor.

# Arbitrage?

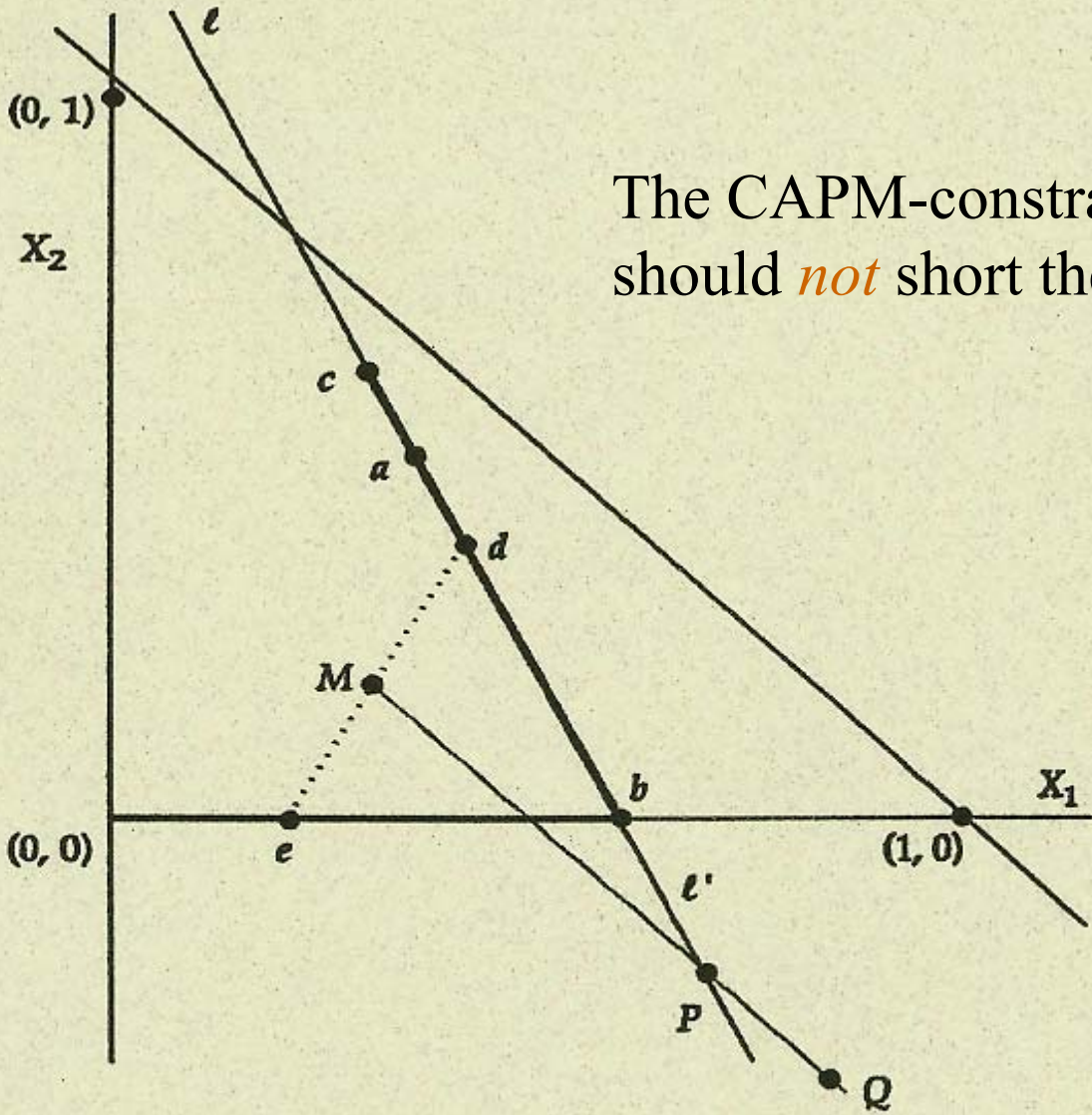
Suppose that most investors are subject to the nonnegativity requirement (2),

but one investor can short, in the CAPM sense.

(Perhaps the CAPM investor has surreptitious access to a vault containing stock certificates that he or she can “borrow” temporarily without posting collateral)

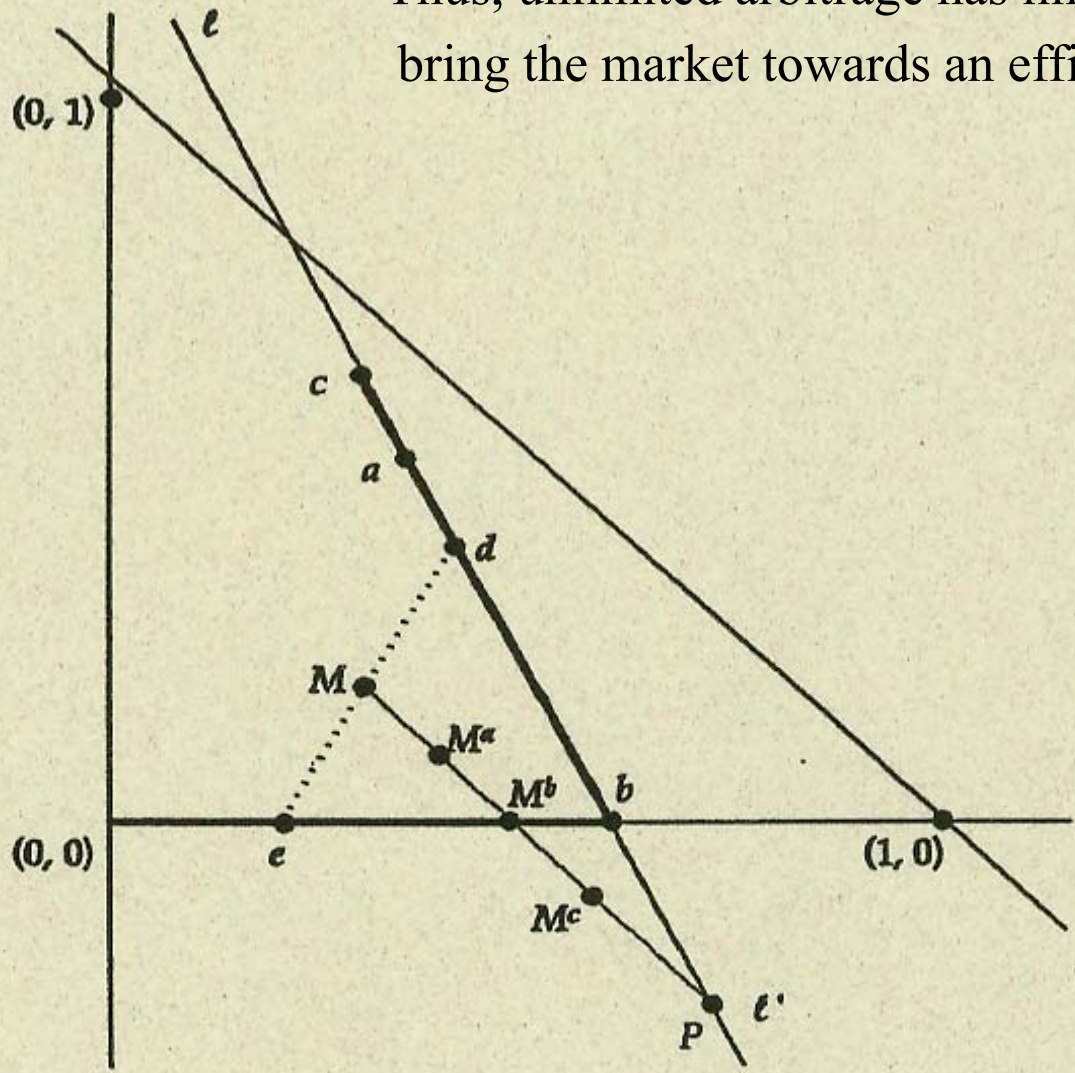
Would this CAPM investor arbitrage away the inefficiency in the market portfolio?

The CAPM-constrained investor should *not* short the market





Thus, unlimited arbitrage has limited power to bring the market towards an efficient portfolio



# Expected Returns and Betas.

If Assumptions 1-4 or 1-4' are true, then Conclusion 2 follows:

Expected returns are linearly related to the betas of each security.

$$E_1 = a + b\beta_1$$

$$E_2 = a + b\beta_2$$

$$E_3 = a + b\beta_3$$

where  $\beta_i$  is the coefficient of regression of the return on the  $i$ th security against the return on the market.

These equations do not necessarily hold if A1-A3 are true but neither A4 nor A4' is true.

**Table 2**  
**Betas vs. P**

Security	Percent in $P$	$cov_{i,P} = P_i V_i$	$beta_{i,P}$
1	0.70%	0.0227	0.52
2	-0.25	-0.0036	- 0.08
3	0.55	0.0495	1.12

Note:  $var(P) = 0.0440$ ;  $Beta_{i,P} = cov_{i,P} / var / P$

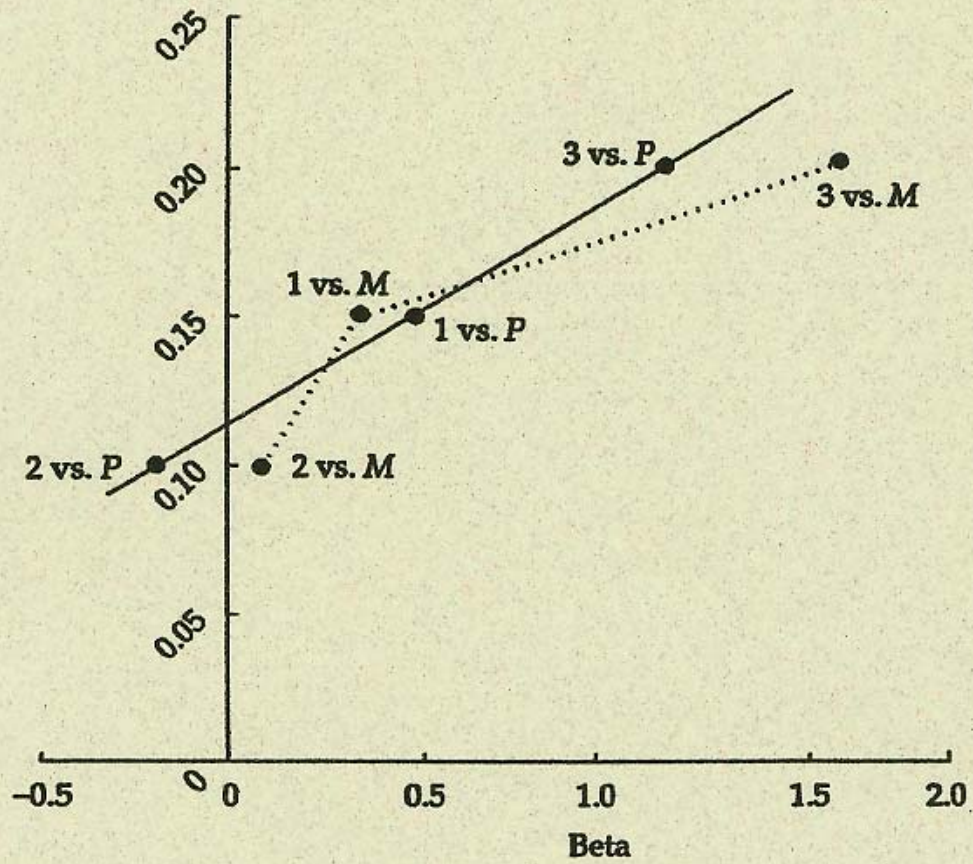
**Table 3**  
**Betas vs. M**

Security	Percent in $M$	$cov_{i,M} = M_i V_i$	$beta_{i,M}$
1	0.30	0.0097	0.36
2	0.19	0.0027	0.10
3	0.51	0.0459	1.71

Note:  $var(M) = 0.0268$ ;  $beta_{i,M} = cov_{i,M} / var(M)$



Expected Return

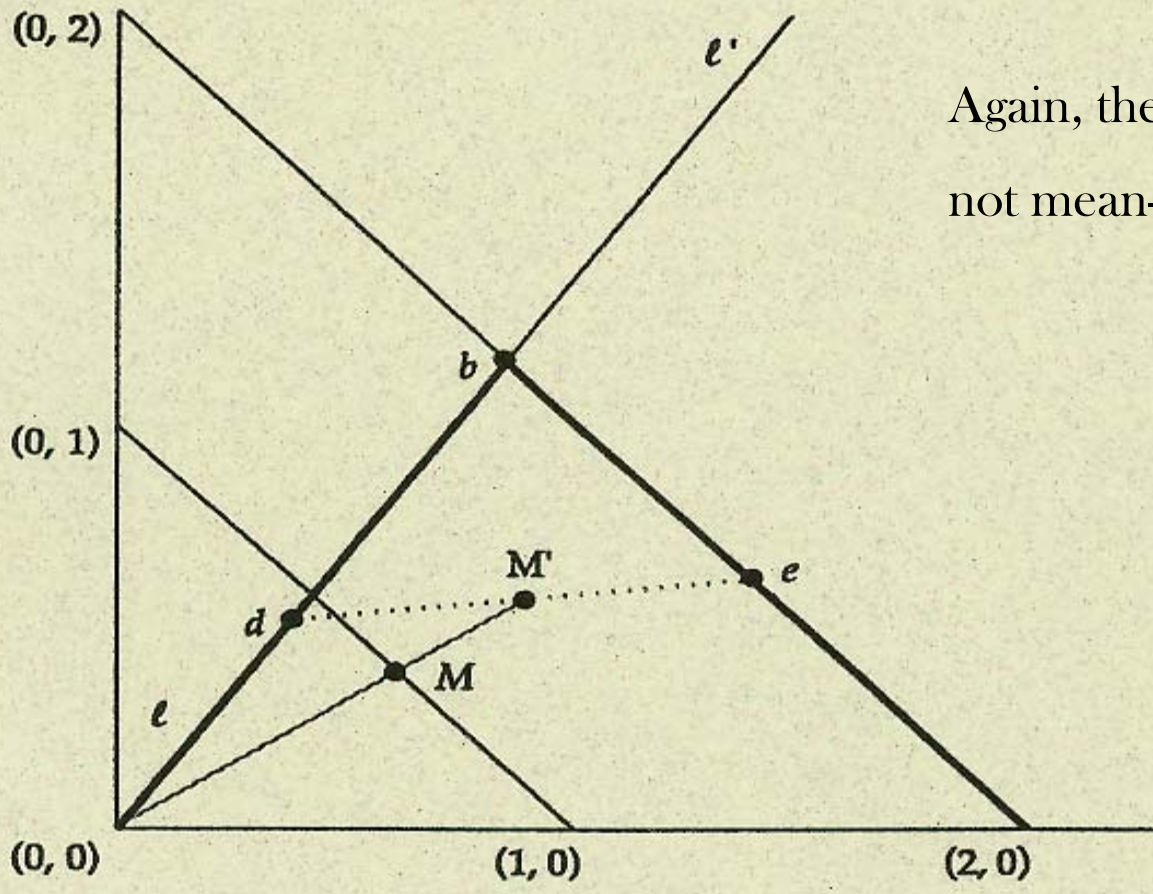




Suppose limited borrowing, e.g. 100%

**Table 4**  
**Example Inputs Including Cash**

Security	Expected Return	Standard Deviation
1	0.15%	0.18%
2	0.10%	0.12%
3	0.03%	0.00%



Again, the market portfolio  $M$  is not mean-variance efficient.

# Generalizations

Suppose that

there are  $n$  securities (for  $n = 3$  or  $30$  or  $3,000$ ),

not all expected returns are the same,

*and*

the  $n$  securities have a nonsingular covariance matrix.

If the only constraint on the choice of portfolio is

$$\sum_{i=1}^n X_i = 1$$

then the portfolios that minimize portfolio variance  $V_P$  for various values of portfolio expected return  $E_P$  lie on a single straight line in  $(n-1)$ -dimensional portfolio space.

Next, consider an investor subject to a no-shorting constraint

$$X_i \geq 0, \quad i = 1, \dots, n$$

The “critical line algorithm” (CLA) begins with the portfolio that is 100 percent invested in the security with highest expected return.

It traces out the set of efficient portfolios in a series of iterations. Each iteration computes one piece (one linear segment) of the piecewise linear efficient set.

Each successive segment has either one more or one less security than the preceding segment.

If the analysis includes a risk-free asset (or, equivalently, risk-free lending), the last segment to be computed (the one with the lowest portfolio mean and variance) is the one and only segment that contains the risk free asset (Tobin 1958)

If the universe consists of, say 10,000 securities, and if all securities are to be demanded by someone, then this universal efficient frontier must contain at least 10,000 segments.

If investors have sufficiently diverse risk tolerances, they will choose portfolios on different segments.

The market portfolio is a weighted average of individual portfolios and typically is not on any efficient segment.

This characterization of efficient sets remains true if limited borrowing is allowed, as illustrated in Figure 5. It also remains true when short selling is permitted but is subject to a Reg T or similar constraint (see Jacobs, Levy, Markowitz 2005)

$$\sum_{i=1}^n X_i + X_{2n+1} - X_{2n+2} = 1,$$

$$\sum_{i=1}^{2n} X_i + X_{2n+3} = 2$$

and

$$X_i \geq 0, \text{ with } i = 1, \dots, 2n+3$$

Portfolio with maximum expected return typically contains two variables at positive levels.

As before, CLA traces out the efficient frontier in a series of iterations--each iteration producing one piece of the piecewise linear efficient set, each piece having one more or (occasionally) one less nonzero variable than did the preceding piece.

If there are 10,000 securities each demanded by someone, the efficient set has at least “10,000” linear segments. Typically the market portfolio is on none of these.



Thus, again, typically

The market portfolio is not mean-variance efficient.

There is not a linear relationship between expected returns and betas.

There is no representative investor

Results presented so far have been negative:

What's *not* true in a NonCAPM world.

Now, something somewhat more positive:

How to compute equilibrium prices in some nonCAPM worlds

Answer:

Use JLMsSim with its (new) CME (Capital Market Equilibrium) modus operandi as opposed to its original DA (Dynamic Analysis) modus operandi.

For example,

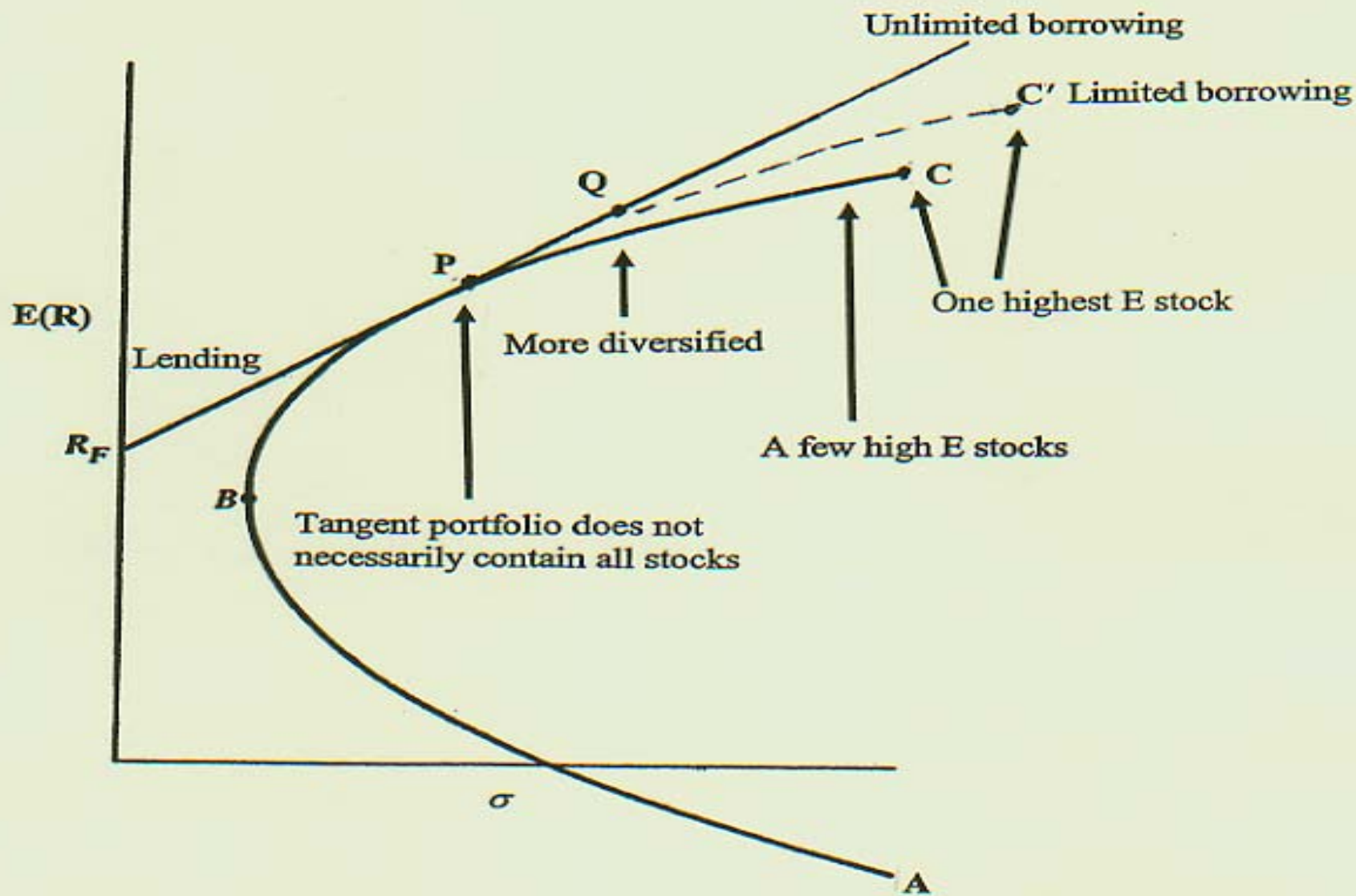
Black-Litterman find expected returns which imply given market weights assuming CAPM (and adjust these according to the users “views” or “prior”).

Problem: CAPM does not have the same equilibrium expected returns as models with real-world constraints.

If CAPM expected returns are assumed for a population of investors subject to real-world short and borrowing constraints then typically

**Their demands for various securities will not add up to the market portfolio.**

The risky assets of cautious investors, who hold cash, will equal the market portfolio. When the holdings of less cautious are added, the totals will not add up to the target weights:



The efficient Frontier, with and without cash and borrowing

Originally, JLMSim was developed as an asynchronous simulator to help think through dynamic consequences of more complex models. All investors in current JLMSim seek mean-variance efficiency.

For a particular simulation run, the JLMSim user specifies:

- How many securities
- How many “investor templates”
- For each investor template:
  - How many investors are to be created from this template
  - How frequently do they reoptimize
  - What  $k$  used in maximizing  $E-kv$ .
  - What probability distribution of initial wealth
  - Random deposits and withdrawals

- How many trader templates
  - What rules do traders of this template use to
    - Set initial bid and asked prices
    - Modify bids and offers, or cancel orders, if order not completed in a timely manner
  
- How many statisticians
  - What rules do they use to estimate means, variances and covariances

Etc.

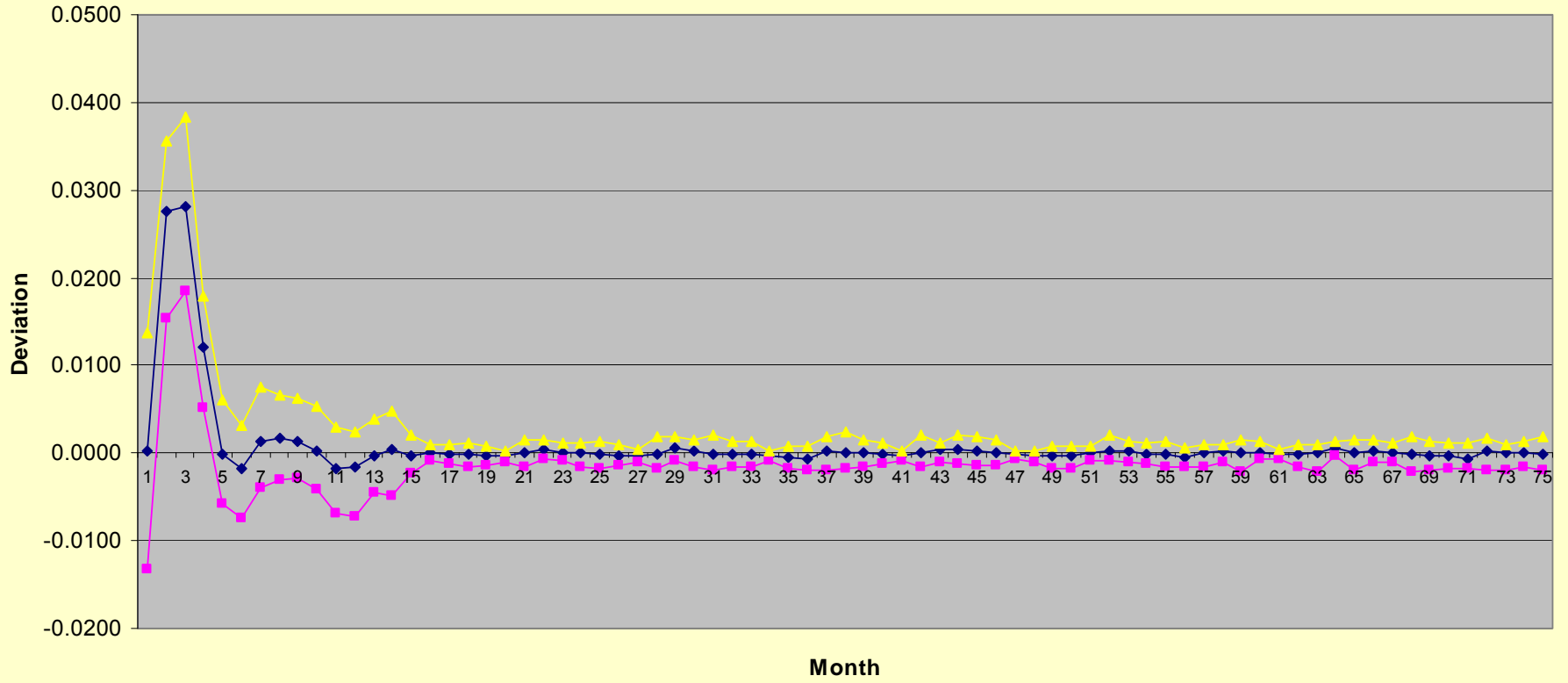
Now, equilibrium expected returns can be computed for markets with real-world constraints by using JLMsim in its CME (Capital Market Equilibrium) rather than its DA (Dynamic Analysis) mode of operation.

With the CME modus operandi the statistician estimates expected return by increasing estimates for securities under target weights, decreasing them for securities over target weights.

The system converges fairly rapidly:



**Max, Min and Avg IntraMonth Deviation From Target Weights for Sec 0  
For Case with 8 "Securities," and Investors with Diverse Risk Aversions**



In sum,

*If* investors cannot borrow all they want at the risk-free rate or short without limit and use the proceeds to buy long positions

*Then*

The market portfolio is not necessarily efficient;  
Expected returns are not a linear function of betas;  
There is no representative investor.

*But*

One can calculate equilibrium expected returns