STATE STREET GLOBAL MARKETS

RESEARCH

Optimal Rebalancing

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Equity Weight in 60/40 Allocation

Outline

> Dynamic programming

> Simplified portfolio rebalancing

> Markowitz-van Dijk heuristic

> Results

Soul Mate Search

- Imagine you have ten years to find a soul mate and you meet one potential soul mate each year.
- You rank each companion on a scale from 0 to 100 and assume that scores are uniformly distributed.
- At the end of each year you must decide to marry your current companion or continue searching.
- You are not allowed to revert to previous companions.
- If you have not found your soul mate by year ten, your parents force you to marry the person you are with at that time.

Years 10 and 9

> The expected score of your companion in year ten is 50.

> Hence, you should marry in year nine only if your companion at the time scores above 50.

Year									9	10
Expected Value										50

Year 8

- There is a 50% chance you will marry your companion in year nine. If you marry in year 9, your companion's expected score is 75 given that it must be above 50 in order for your companion to be marriageable.

- Your hurdle for year 8 is 50% x 75 + 50% x 50 = 62.5. You should marry your current companion only if he or she scores above 62.5

Year									9	10
Expected Value									62.5	50

Years 1-7

> The likelihood that your companion in year eight will score above 62.5 is 37.5%. The expected score of this marriageable companion is 81.25.

> Your hurdle for year 7 is 37.5% x 81.25 + 62.5% x 62.5 = 69.5.

> By proceeding in this fashion we determine the scores for each year.

Year	1	2	3	4	5	6	7	8	9	10
Expected Value	86.1	85	83.6	82.0	80.0	77.5	74.2	69.5	62.5	50

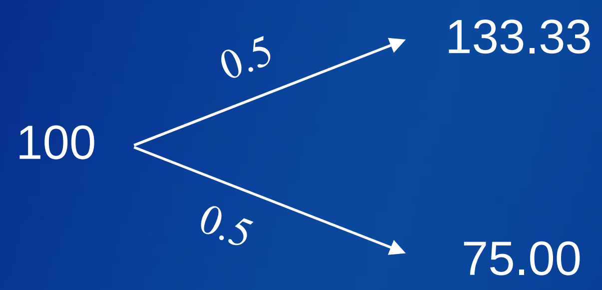
Simplified Portfolio Rebalancing

Optimal portfolio: 60% Stocks, 40% Bonds.

Probability	Stock Return	Bond Return
25%	26.00%	1.00%
50%	8.00%	8.00%
25%	-11.00%	10.00%

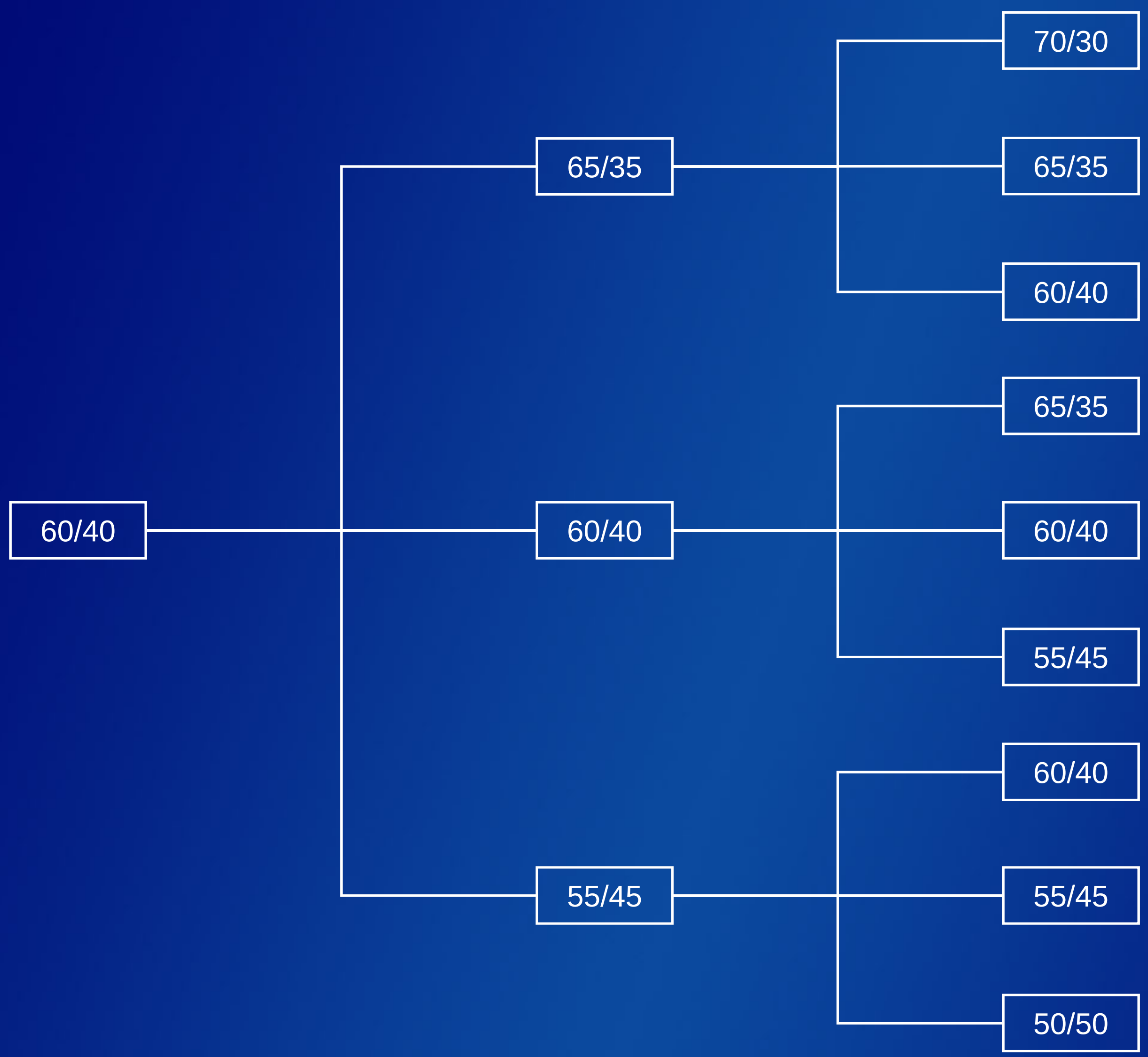
Sub-optimality Cost

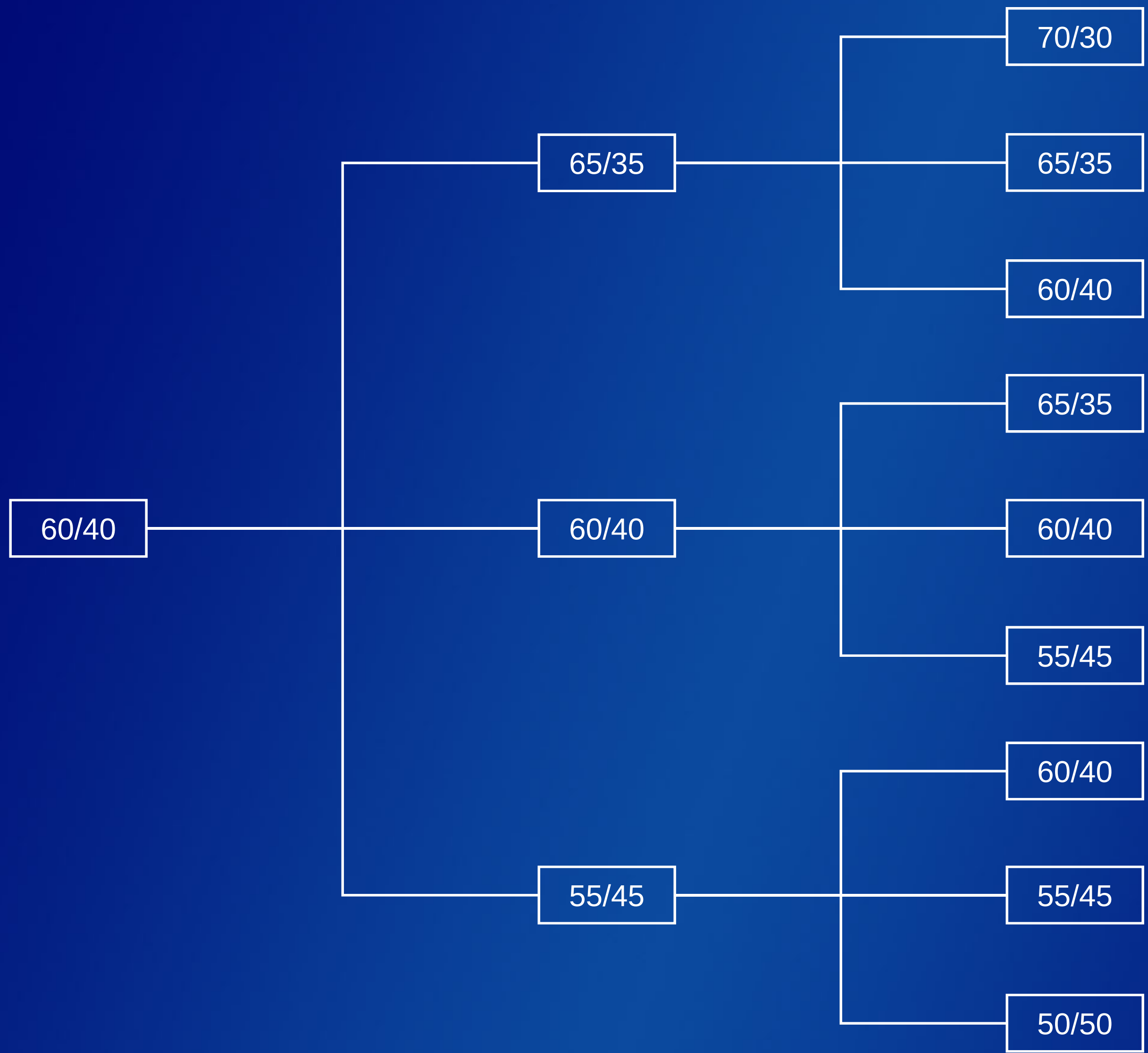
> Consider a $100 gamble that has an equal chance of increasing by 1/3 or decreasing by 1/4.



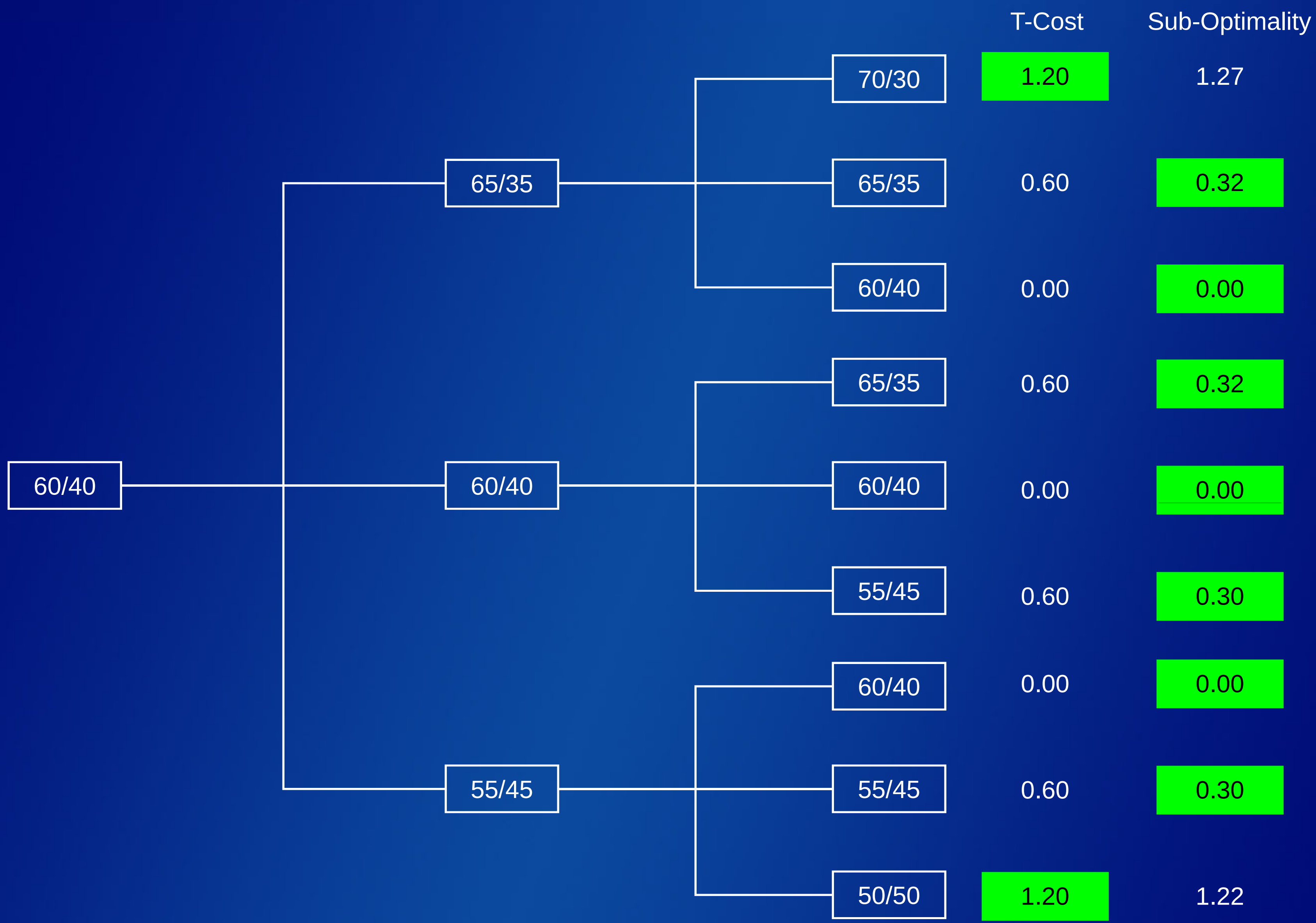
> For a log wealth investor, expected utility equals: ln(133.33) x .5 + ln(75.00) x .5 = 4.60517

> The ln(100.00) also equals 4.60517. Therefore, 100.00 is the certainty equivalent of a risky gamble that has an equal chance of yielding 133.33 or 75.00.

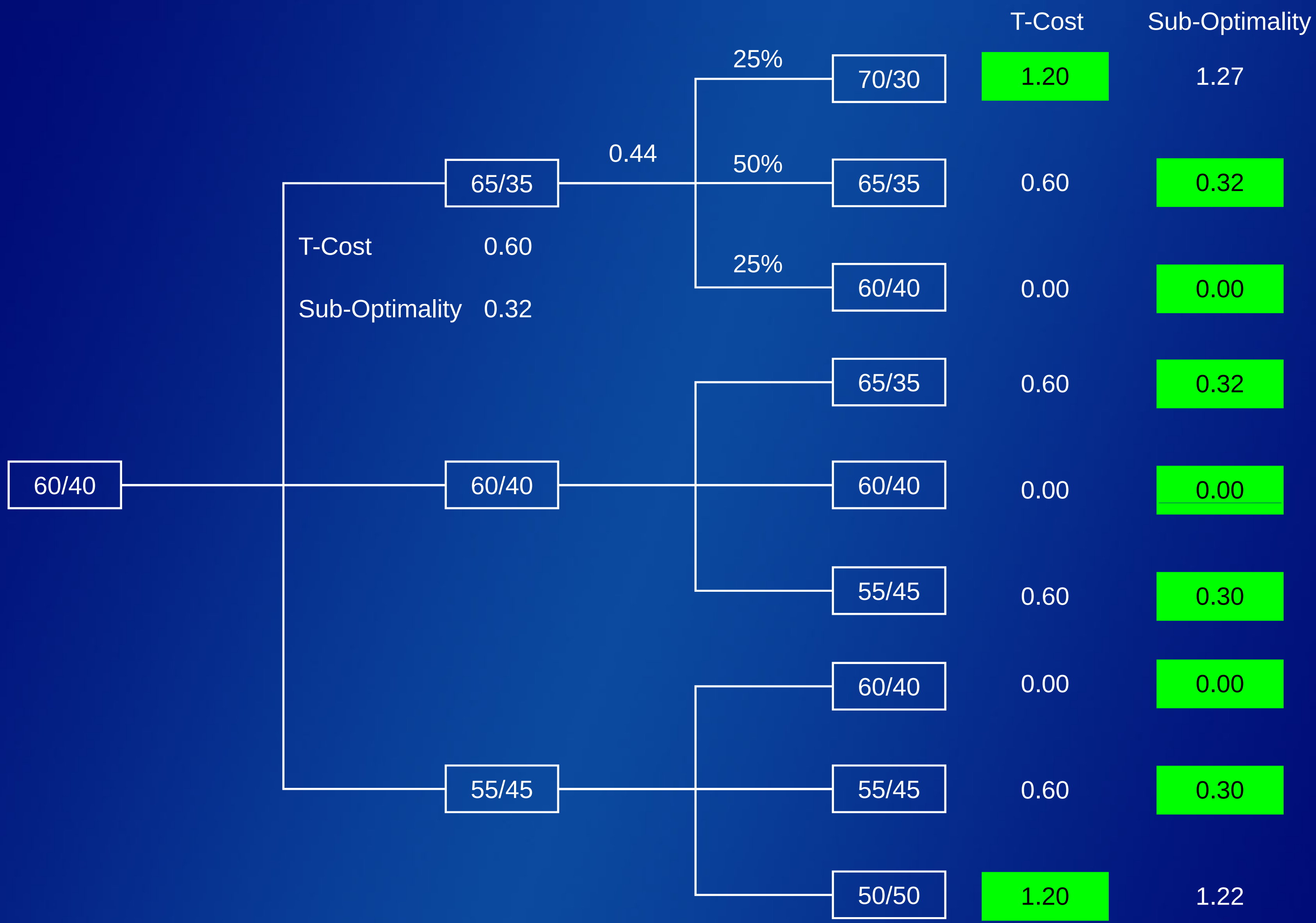

60/40
65/35
60/40
55/45
70/30
65/35
60/40
65/35
60/40
55/45
60/40
55/45
50/50



	T-Cost	Sub-Optimality
70/30	1.20	1.27
65/35	0.60	0.32
60/40	0.00	0.00
65/35	0.60	0.32
60/40	0.00	0.00
55/45	0.60	0.30
60/40	0.00	0.00
55/45	0.60	0.30
50/50	1.20	1.22


T-Cost
Sub-Optimality
60/40
65/35
60/40
55/45
70/30
1.20
1.27
65/35
0.60
0.32
60/40
0.00
0.00
65/35
0.60
0.32
60/40
0.00
0.00
55/45
0.60
0.30
60/40
0.00
0.00
55/45
0.60
0.30
50/50
1.20
1.22


T-Cost
Sub-Optimality
60/40
65/35
T-Cost 0.60
Sub-Optimality 0.32
60/40
55/45
70/30 1.20 1.27
65/35 0.60 0.32
60/40 0.00 0.00
65/35 0.60 0.32
60/40 0.00 0.00
55/45 0.60 0.30
60/40 0.00 0.00
55/45 0.60 0.30
50/50 1.20 1.22


T-Cost
Sub-Optimality
60/40
65/35
T-Cost 0.60
Sub-Optimality 0.32
0.44
25%
70/30
1.20
1.27
50%
65/35
0.60
0.32
25%
60/40
0.00
0.00
60/40
65/35
0.60
0.32
60/40
0.00
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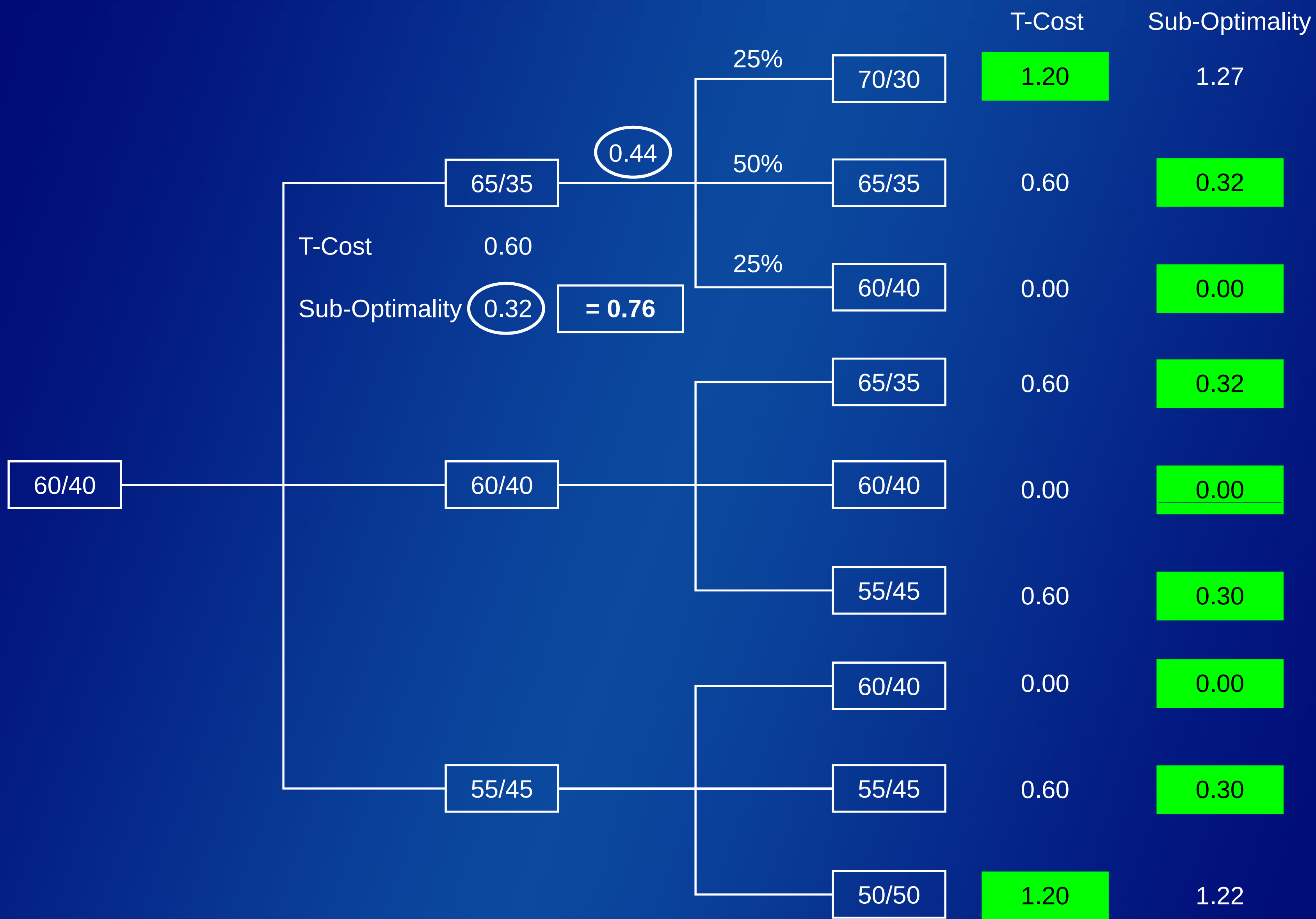
55/45
0.60
0.30
55/45
60/40
0.00
0.00
55/45
0.60
0.30
50/50
1.20
1.22


T-Cost
Sub-Optimality
25%
70/30
1.20
1.27
0.44
50%
65/35
65/35
0.60
0.32
T-Cost
0.60
25%
60/40
0.00
0.00
Sub-Optimality
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= 0.76
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50/50
1.20
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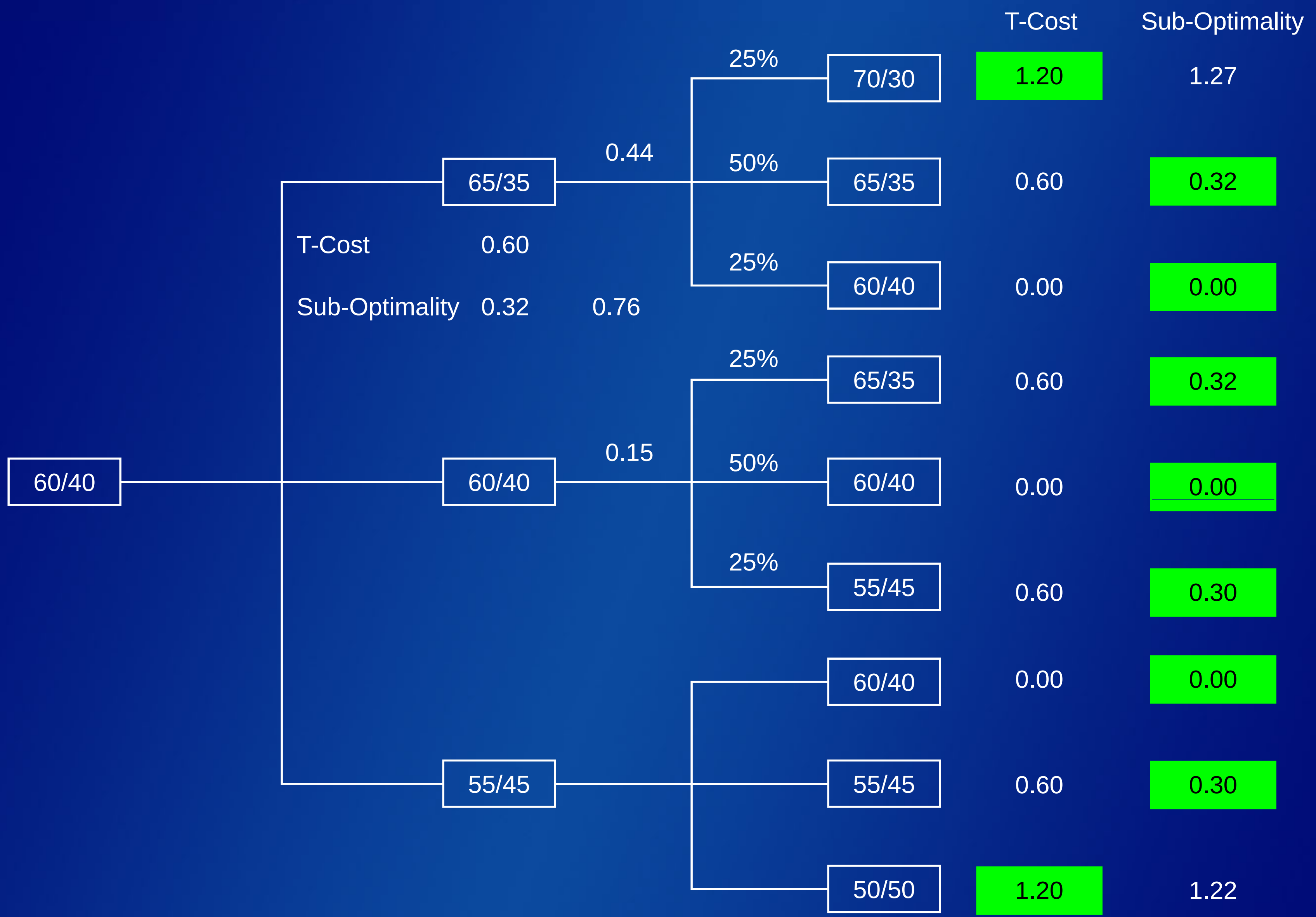
T-Cost
Sub-Optimality
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T-Cost 0.60
Sub-Optimality 0.32 0.76
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70/30 1.20 1.27
65/35 0.60 0.32
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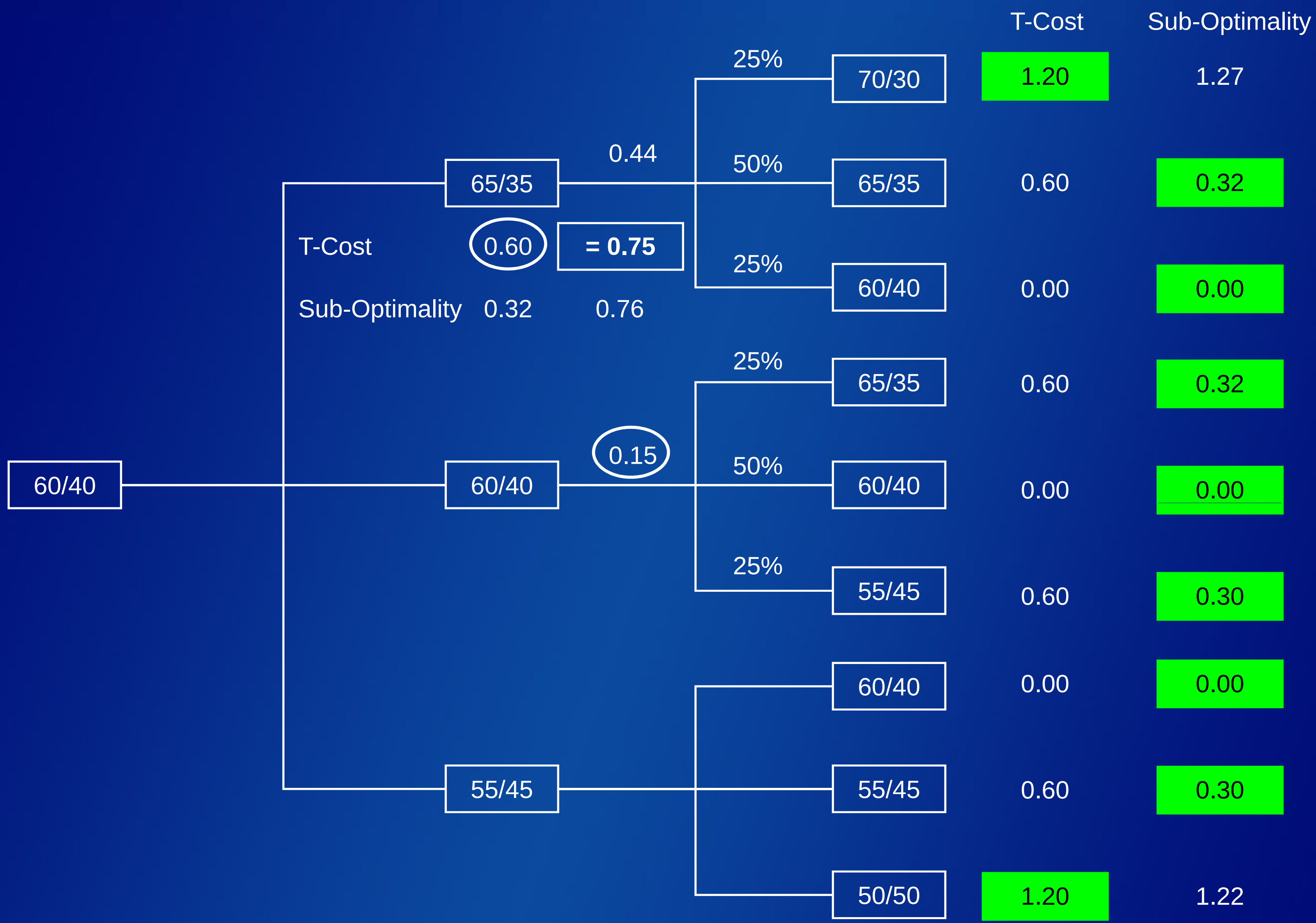
55/45 0.60 0.30
50/50 1.20 1.22


T-Cost
Sub-Optimality
60/40
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0.44
T-Cost 0.60 = 0.75
Sub-Optimality 0.32 0.76
60/40
0.15
55/45
25%
70/30
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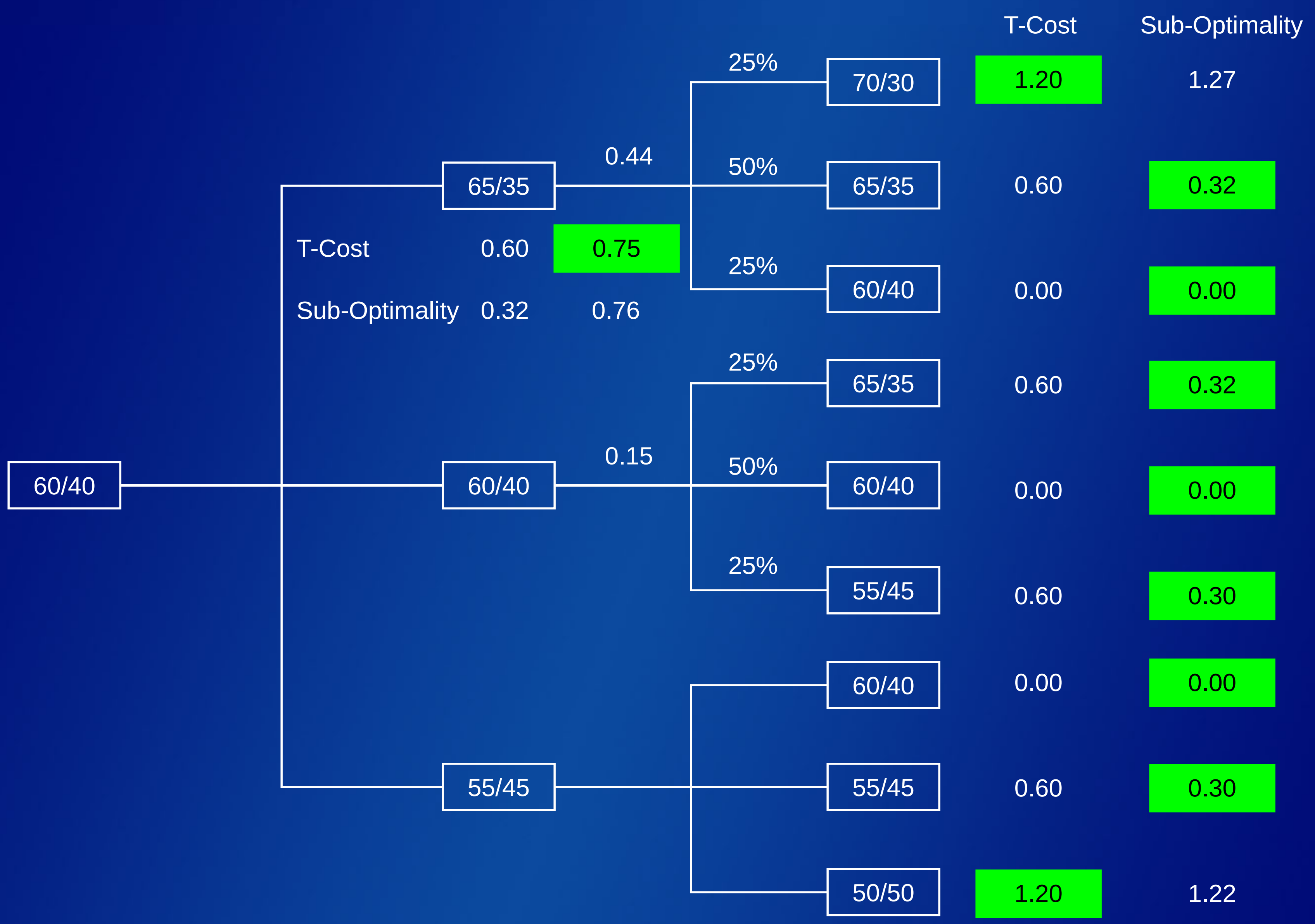
50/50
1.20
1.22


T-Cost
Sub-Optimality
60/40
65/35
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55/45
0.44
0.15
T-Cost 0.60 0.75
Sub-Optimality 0.32 0.76
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50%
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70/30 1.20 1.27
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60/40 0.00 0.00
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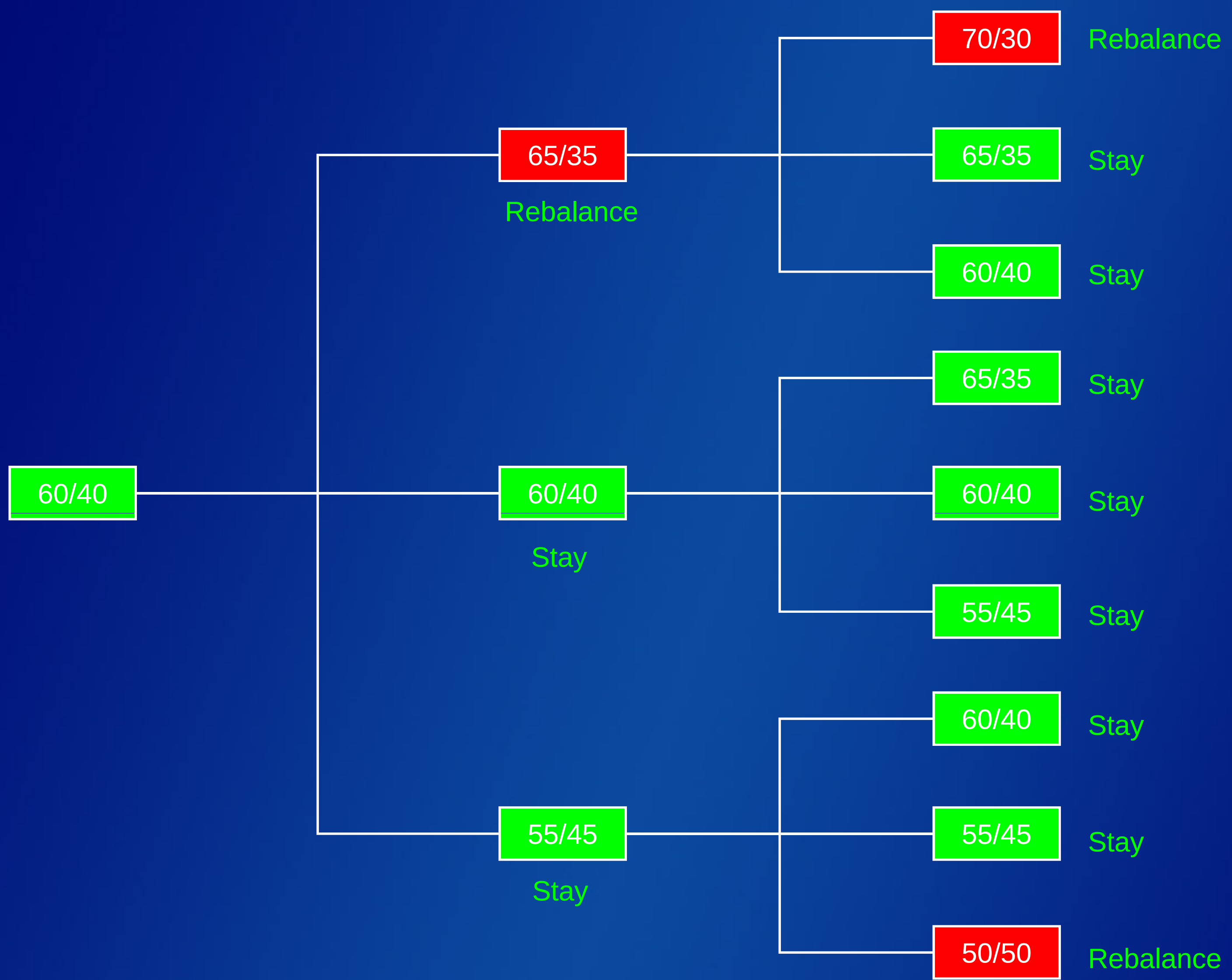
60/40 0.00 0.00
55/45 0.60 0.30
60/40 0.00 0.00
55/45 0.60 0.30
50/50 1.20 1.22


60/40
65/35
Rebalance
60/40
Stay
55/45
Stay
70/30
Rebalance
65/35
Stay
60/40
Stay
65/35
Stay
60/40
Stay
55/45
Stay
60/40
Stay
55/45
Stay
50/50
Rebalance

The Curse of Dimensionality

As we add more assets, increase time horizon, increase granularity, and allow for partial rebalancing, the computational challenge rises sharply.

Number of Asset	Number of Portfolios	Number of Calculations to Perform
2	101	5,620,751
3	5,151	14,619,573,351
4	176,851	17,233,228,186,751
5	4,598,126	11,649,662,254,243,700
6	96,560,646	5,137,501,054,121,460,000
7	1,705,904,746	1,603,471,162,336,350,000,000
8	26,075,972,546	374,655,945,665,079,000,000,000
9	352,025,629,371	68,281,046,097,460,800,000,000,000
10	4,263,421,511,271	10,015,396,403,505,300,000,000,000,000

*12 time periods, 1% granularity.

The Markowitz-van Dijk Heuristic

$$E(U) = \sum_{i=1}^{m} p_i \ln\left(1 + \sum_{j=1}^{n} X_j \mu_{ij}\right) = p \ln(1 + \mu X')$$

$$\mu = \begin{bmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1n} \\ \mu_{21} & \mu_{22} & \cdots & \mu_{2n} \\ & & \vdots & \\ \mu_{m1} & \mu_{m2} & \cdots & \mu_{mn} \end{bmatrix}$$

$$X = [X_1, \ldots, X_n] \qquad p = [p_1, \ldots, p_m]$$

$$J_t(X_t, X_{t-1}) = e^{\ln\left(1+\sum_{j=1}^{n} X_j^{opt} \mu_j\right)} - e^{\ln\left(1+\sum_{j=1}^{n} X_{jt} \mu_j\right)} + \sum_{i=1}^{n} C_j \left| X_{jt} - X_{jt-1} \right| + J_{t+1}(X_{t+1}, X_t)$$

$$J_t(X_t, X_{t-1}) = e^{\ln(1+X^{opt}\mu')} - e^{\ln(1+X_t \mu')} + C\left| X_t - X_{t-1} \right| + J_{t+1}(X_{t+1}, X_t)$$

$$J_t(X_t, X_{t-1}) = e^{\ln\left(1+\sum_{j=1}^{n} X_i^{opt} \mu_j\right)} - e^{\ln\left(1+\sum_{j=1}^{n} X_{it} \mu_i\right)} + \sum_{i=1}^{n} C_j \left| X_{jt} - X_{jt-1} \right| + \sum_{i=1}^{n} d_i \left(X_i - X^{opt}{}' \right)^2$$

How to Solve for *d*

- We generate 200 possible incoming portfolios given the expected returns, variances, and covariances of the component assets of the initial optimal portfolio along with its weights.

- For a given coefficient d, we solve for a new portfolio for each of the incoming portfolios such that we minimize cost as defined by:

$$J_t(X_t, X_{t-1}) = e^{\ln\left(1+\sum_{j=1}^{n} X_i^{opt}\mu_j\right)} - e^{\ln\left(1+\sum_{j=1}^{n} X_{it}\mu_i\right)} + \sum_{i=1}^{n} C_j \left| X_{jt} - X_{jt-1} \right| + \sum_{i=1}^{n} d_i \left(X_i - X^{opt\prime} \right)^2$$

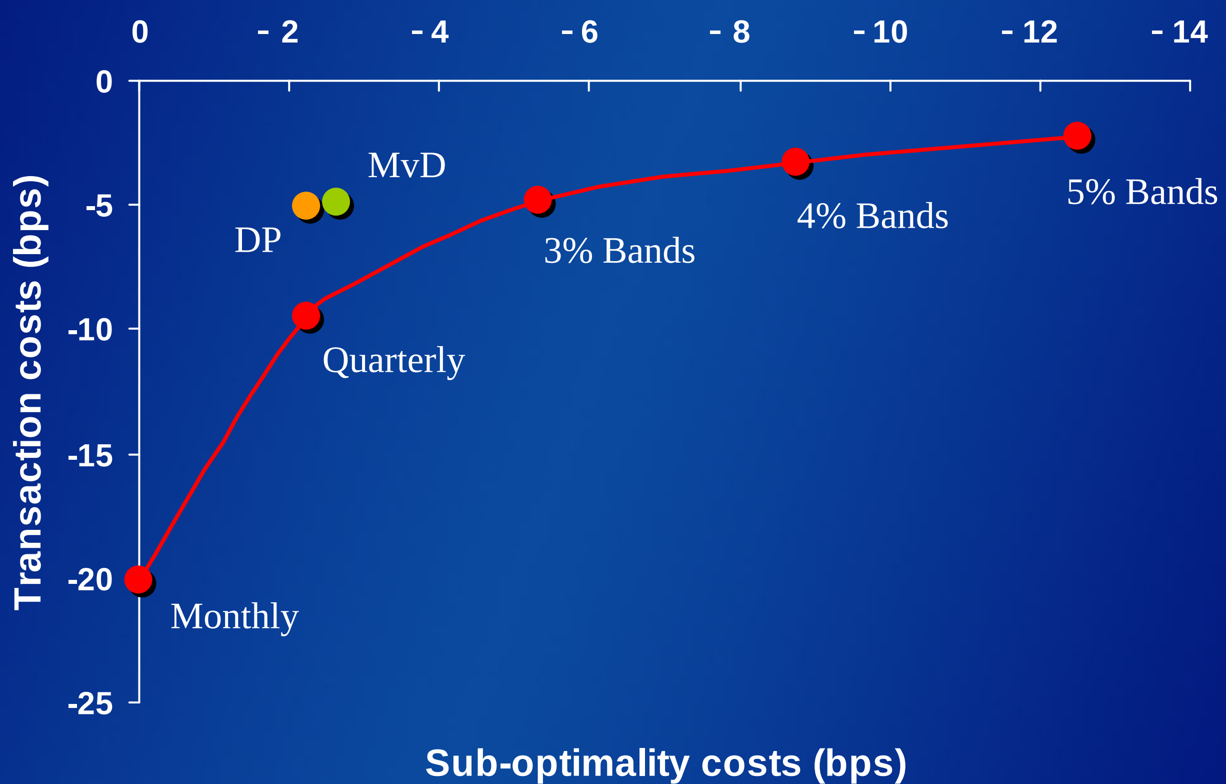
- We proceed forward through 12 periods and accumulate the costs. We then calculate a figure of merit by taking the average of the 200 cumulative costs.

- Next we select a new value for the coefficient d and repeat the process. We proceed in this fashion until we identify the coefficient which produces the best figure of merit.

Results

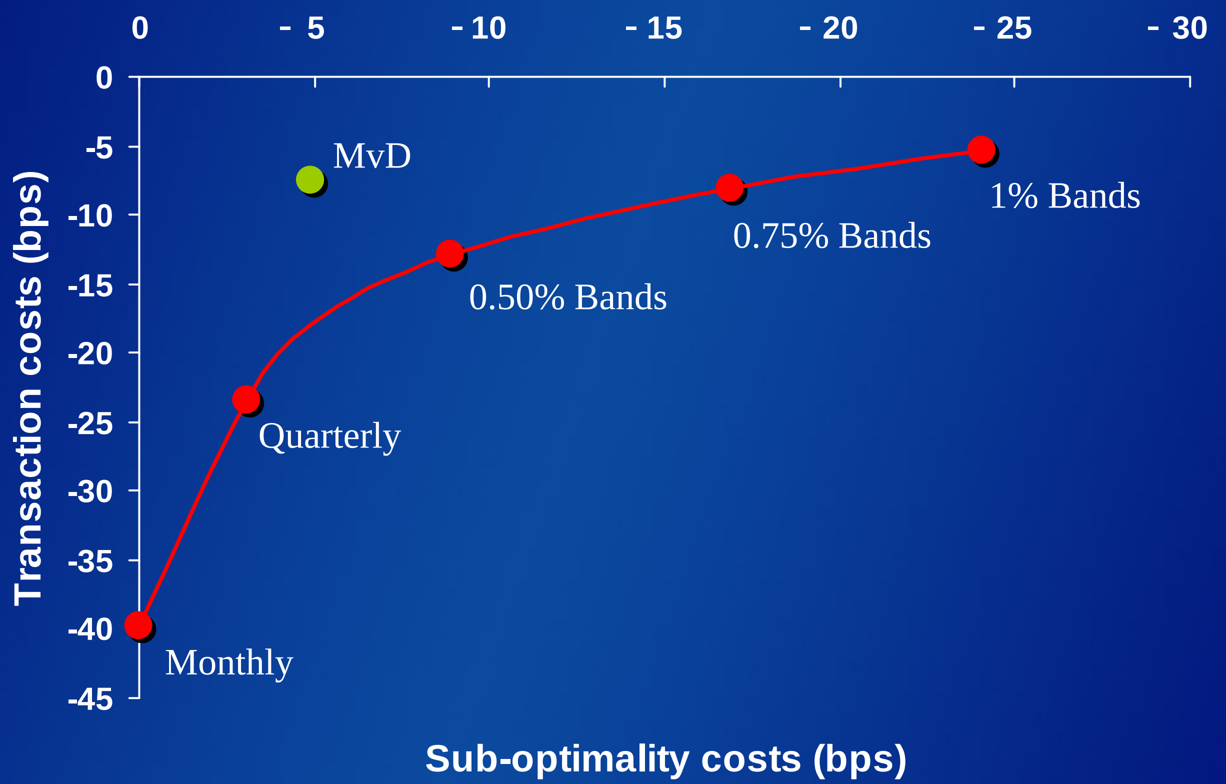
Four Assets

(40% US Equity, 25% US Bonds, 20% Non-US Equity, 15% Non-US Bonds)



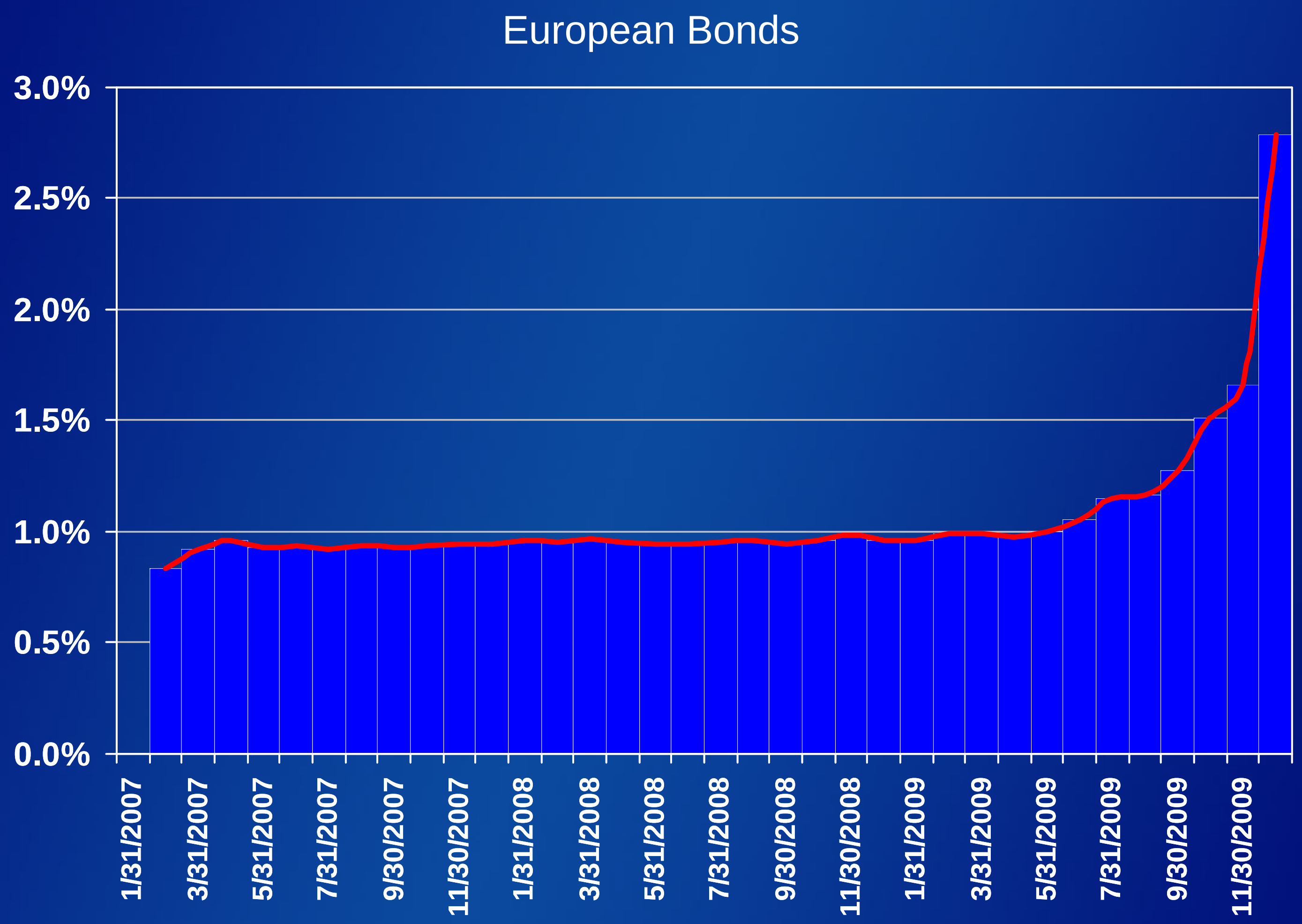
One Hundred Assets

(100 securities selected from the S&P 500)

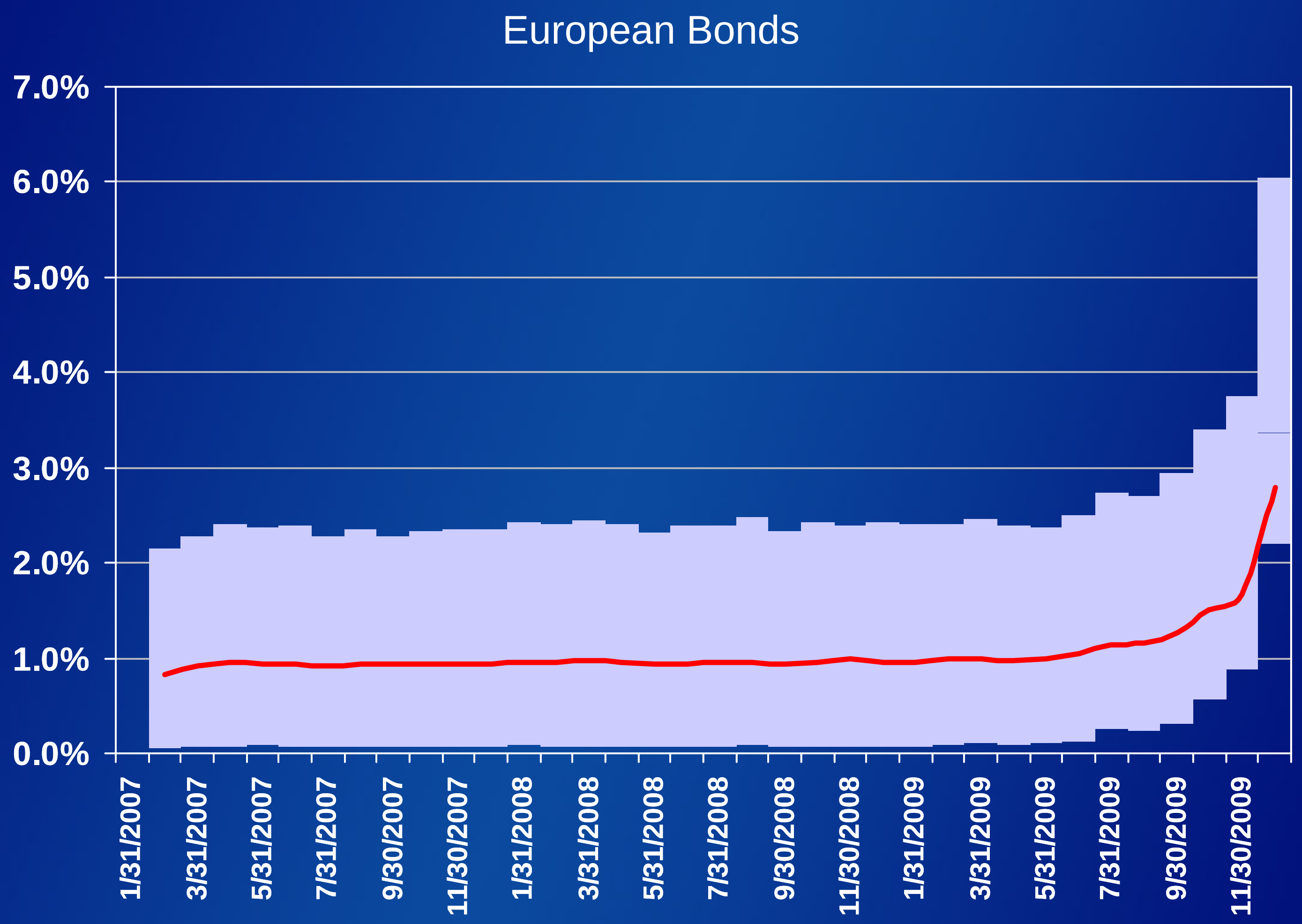


Rebalance Trigger (Average)

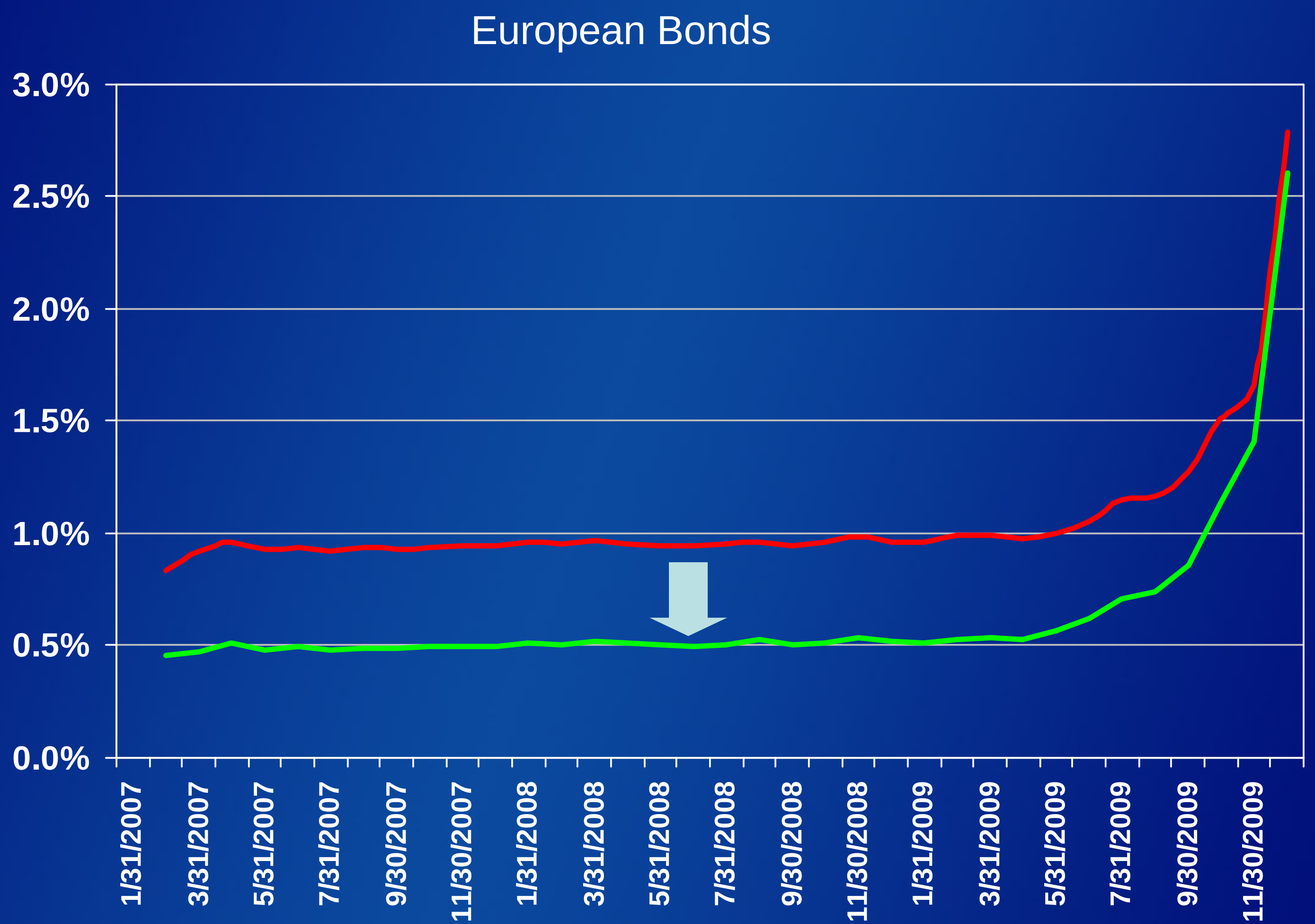
(65% European Bonds, 25% Foreign Equity, 10% Domestic Equity)



Distribution



Average Trade



Transaction Cost Savings

Number of Assets	Approach	Optimal Rebalancing Costs (bps)	Industry Heuristics Average Costs (bps)	Total Savings
2	Dynamic Programming	6.31	10.91	42%
3	Dynamic Programming	6.66	10.97	39%
4	Dynamic Programming	7.33	13.28	45%
5	MvD Heuristic	8.61	14.02	39%
100	MvD Heuristic	12.46	27.70	55%

Transaction Cost Savings

Number of Assets	Closest Heuristic*	Trading Costs (bps)	Optimal Strategy Trading Costs (bps)	Savings on Trading Costs	Annual Savings $5 billion Portfolio
2	2% Bands	7.18	4.87	32%	$1,155,000
3	3% Bands	5.40	4.68	13%	$360,000
4	2% Bands	7.29	5.10	30%	$1,095,000
5	2% Bands	7.70	6.21	19%	$745,000
100	Semi-Annually	16.64	7.55	55%	$4,545,000

* Chosen as the strategy with the same or slightly higher tracking error risk.

Success Rates

Rebalancing Strategy	Optimal Rebalancing	2% Bands	Daily	Variable Bands
Optimal Rebalancing		96.40%	99.30%	99.60%
		20.21 bps	25.51 bps	25.28 bps
2% Bands	3.60%		66.90%	54.90%
	3.42 bps		16.08 bps	49.39 bps
Daily	0.70%	33.10%		62.80%
	4.81 bps	14.55 bps		43.67 bps
Variable Bands	0.40%	45.10%	37.20%	
	0.72 bps	9.80 bps	18.59 bps	

Summary

- In an idealized world without transaction costs investors would rebalance continually to the optimal weights. In the presence of transaction costs investors must balance the cost of sub-optimality with the cost of restoring the optimal weights.
- Most investors employ heuristics that rebalance the portfolio as a function of the passage of time or the size of the misallocation.
- We employ multi-period optimization to determine optimal rebalancing rules, and we demonstrate that this approach is significantly superior to standard industry heuristics.