Optimal Asset Allocation and Risk Shifting in Money Management*

Suleyman Basak
London Business School and CEPR
Regents Park
London NW1 4SA
United Kingdom
Tel: 44 (0)20 7706-6847
Fax: 44 (0)20 7724-3317
E-mail: sbasak@london.edu

Anna Pavlova
Sloan School of Management
Massachusetts Institute of Technology and CEPR
50 Memorial Drive, E52-433
Cambridge, MA 02142-1347
Tel: (617) 253-7159
Fax: (617) 258-6855
E-mail: apavlova@mit.edu

Alex Shapiro
Stern School of Business
New York University
Fax: (212) 995-4233
E-mail: ashapiro@stern.nyu.edu

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Abstract

Money managers are rewarded for increasing the value of assets under management, and predominantly so in the mutual fund industry. This gives the manager an implicit incentive to exploit the well-documented positive fund-flows to relative-performance relationship by manipulating her risk exposure. In a dynamic portfolio choice framework, we show that the ensuing convexities in the manager's objective give rise to a finite risk-shifting range over which she gambles to finish ahead of her benchmark. Such gambling entails either an increase or a decrease in the volatility of the manager's portfolio, depending on her risk tolerance. In the latter case, the manager reduces her holdings of the risky asset despite its positive risk premium. Our empirical analysis lends support to our main insights. Under multiple sources of risk, with both systematic and idiosyncratic risks present, we show that optimal managerial risk shifting may not necessarily involve taking on any idiosyncratic risk. Costs of misaligned incentives to investors resulting from the manager's policy are demonstrated to be economically significant.

JEL Classifications: G11, G20, D60, D81.

Keywords: Fund Flows, Implicit Incentives, Risk Taking, Relative Performance, Risk Management, Portfolio Choice.

1. Introduction

"The real business of money management is not managing money, it is getting money to manage."[1] Indeed, with the number of mutual funds in the US exceeding the number of stocks, fund managers are increasingly concerned with attracting investors' money. Recent empirical evidence (e.g., Gruber (1996), Chevalier and Ellison (1997), Sirri and Tufano (1998)), offers simple insight to a manager: new money is expected to flow into the fund if the manager has performed well relative to a certain benchmark. With her compensation typically increasing in the value of assets under management, this positive fund-flows to relative-performance relationship creates an *implicit incentive* for the manager to increase the likelihood of future fund inflows, distorting her asset allocation choice. There is, of course, also an *explicit incentive* induced by the manager's compensation: managing assets in line with her own appetite for risk, which need not coincide with those of fund investors. This is another source of conflict between a fund manager and her investors, originally pointed out by Ross (1973).[2] Together, the manager's implicit and explicit incentives shape her asset allocation policy. Understanding this policy is of utmost importance to fund investors who may be hurt by adverse incentive effects. Our objective is to examine the implications of these incentives within a familiar dynamic portfolio choice framework.

We consider a dynamic economy and focus on a fund manager, guided by a risk-averse objective. The manager's compensation depends on the total value of the fund at some terminal date (e.g., end of the year). This fund value is determined by the portfolio choice of the manager during the year and by nonmarketable inflows/outflows of new money at the year-end. The rate of flows into the fund depends on the manager's performance over the year relative to a benchmark − a reference portfolio of stock and money markets. We consider several types of the flow-performance relationship, with the baseline specification being adopted from Chevalier and Ellison.[3] These specifications share the common feature that they all give rise to local convexities in the manager's objective function. The presence of such local convexities and their importance have been noted by numerous studies. In our analysis, we uncover several novel implications entailed by convexities that are robust across all specifications of the manager's payoff, and are at odds with conventional views expressed in the literature.

The manager's optimal policy reflects the interplay between the implicit incentives to boost

[1] As eloquently put by Mark Hurley in the famous Goldman, Sachs and Co. report on the evolution of the investment management industry (see WSJ 11/16/95 and e.g., http://assetmag.com/story/20010601/10438.asp).

[2] Ross argues that an agency conflict arising due to differences in utility functions when the agent's action is difficult to monitor is the most fundamental delegation problem. This problem is especially relevant in the context of money management, where investors cannot frequently monitor the trading strategies of a fund. Of course, the distinction between implicit and explicit incentives becomes blurred when the flow-performance relationship is reinterpreted as part of an explicit compensation contract of a manager.

[3] We attempt to reinterpret some of these specifications as forms of managerial compensation contracts (Section 2.3). Our examples include the so-called 80/120 plans offered to US executives and the asymmetric fee structures, common amongst the European mutual funds.

risk induced by these convexities and her "normal," absent implicit incentives, policy driven by her attitude towards risk. We find that such a tradeoff gives rise to a "risk-shifting" range over which the manager deviates significantly from her normal policy by taking on additional risk (in line with Jensen and Meckling (1976)). Our analysis, however, identifies two possible directions in which the manager optimally manipulates her risk exposure in this range: by boosting her portfolio volatility or by decreasing it. The latter direction is somewhat unexpected: how can "gambling" to finish ahead of the benchmark be consistent with a decrease in portfolio volatility? Simply, in the context of relative performance evaluation, any strategy entailing a deviation from the benchmark is inherently risky. By taking on more systematic risk than that of the benchmark (boosting portfolio volatility), the manager gambles to improve her relative standing when the benchmark goes up. Similarly, by taking on less systematic risk than that of the benchmark (reducing volatility), the manager bets on improving her relative performance when the benchmark falls. The direction of the manager's choice of a deviation from the benchmark depends on her risk aversion: a more risk tolerant manager decides to boost volatility, while a relatively risk averse manager does the opposite, despite the positive risk premium offered by the risky asset.

Since the manager is risk averse, the range over which she engages in excessive risk taking, as well as her ensuing risk exposure, are finite. No risk shifting takes place when the manager is considerably behind or ahead of the benchmark, where the flows-induced incentives are weak and the normal policy considerations prevail.[4] Overall, the strength of managerial risk-taking incentives is highly time and state-dependent, with a maximum positioned somewhere deep in the underperformance region and a minimum differing across the specifications of the flow-performance relationship we consider. These implications are typically in disagreement with the predictions of extant work arguing that risk taking is most pronounced next to convex kinks or discontinuities in a manager's payoff (induced by implicit incentives or explicit compensation contracts; see, e.g., Chevalier and Ellison (1997), Murphy (1999)). In fact, in one of our specifications, we demonstrate that the risk-taking incentives are actually minimized around a discontinuity. These sharp differences in implications are due to the differences in adopted measures of risk taking. The traditional definition of a risk-taking incentive (e.g., standard in corporate finance) is the sensitivity of the manager's payoff to volatility. This measure captures the strength of the manager's desire to increase her risk exposure relative to some fixed status quo asset allocation. Our measure of risk taking is the optimal risk exposure as defined in the portfolio choice literature: the fraction of the fund optimally invested in the risky asset. Intuitively, instead of just taking a partial derivative of the manager's value function with respect to volatility, we take this derivative and equate it to zero to derive the optimal volatility for each level of relative performance.

For most of our analysis we adopt the simplest possible setting, the Black and Scholes (1973)

[4]Some flow-performance specifications that we consider alter the effective risk aversion of the manager. The limits to which the optimal risk exposure converges then accordingly reflect that (Section 2.3).

economy with a single source of risk, to convey our most important insights pertaining to managerial risk shifting. To investigate the manager's portfolio allocation across different stocks and exposure to systematic versus idiosyncratic risk, we extend our baseline model to multiple sources of uncertainty. The overall behavior of the manager is analogous to that in the baseline analysis. However, the manager's incentive to deviate from the benchmark portfolio when underperforming now manifests itself not in increasing or decreasing the fund's volatility but in tilting the weight in each risky security away from the benchmark. We also demonstrate that risk shifting does not necessarily involve taking on idiosyncratic risk. Indeed, when faced with both systematic and idiosyncratic risks, the manager may very well optimally expose herself to no idiosyncratic risk, while engaging in her optimal risk shifting via systematic risk only.

Costs of misaligned incentives resulting from the manager's policy are shown to be economically significant.[5] We compare the manager's policy when acting in the best interest of fund investors, fully ignoring her explicit and implicit incentives, with when accounting for incentives optimally (discussed above), under our baseline flow for relative-performance specification. The difference is quantified in units of an investor's initial wealth. For example, if the investor's relative risk aversion is 2 and the manager's is 1, we find the cost to investor to be nearly 10% of his initial wealth. We demonstrate how the manager's explicit and implicit incentives reinforce each other. The cost due to explicit incentives is particularly severe when the manager's and investor's attitudes towards risk differ substantially, and the cost due to implicit incentives is particularly high when the flow-performance reward is high or when the benchmark is very risky.

Finally, we collect daily returns of US mutual funds to examine empirically the testable implications of our model. Since we are not the first to study risk taking of US fund managers, we focus on testing the implications that are novel to our model.[6] Our first hypothesis is that underperforming managers boost the deviation of their portfolio from the benchmark, the tracking error variance. We find support for this hypothesis, and no evidence that underperforming managers increase the volatility of their portfolios. The latter result is consistent with Busse (2001). The second set of tests examines portfolio betas of underperforming managers. The hypothesis of interest is whether managers who are sufficiently risk averse decrease their portfolio beta (their risk exposure) when underperforming the market. We find that managers who chose lower risk portfolios in a previous year, decrease their portfolio betas in the subsequent year when underperforming. Also, when we

[5]While we focus on the costs arising from misalignment of objectives, there could also be benefits to hiring a manager. After all, the investors have chosen active management. For example, the manager might have lower transactions, market participation or informational costs (Merton (1987)), lower opportunity cost of time (time constraints and other forms of investors' bounded rationality (Rubinstein (1998)), better investments-specific education, or better information or ability. Modeling the benefits together with the costs requires a full-fledged model of interaction between managers and investors, which is beyond the scope of this paper.

[6]Empirical work on risk taking by US fund managers includes Brown, Harlow, and Starks (1996), Chevalier and Ellison (1997), Busse (2001), and more recently, Reed and Wu (2005). Reed and Wu present a broad set of tests consistent with the main predictions of this paper, and argue that risk-shifting behavior of mutual fund managers is due to the benchmark- and not tournaments-induced incentives.

examine the subsample of funds whose current year's betas are below unity, we again find that these funds decrease beta when underperforming the market. However, while the signs of these responses are consistent with our theory, their magnitudes are considerably smaller, most likely reflecting the risk-management constraints imposed in practice (see Almazan, Brown, Carlson, and Chapman (2004)).

Related to our work is the literature examining implicit incentive conflicts in money management. To study flows-induced risk taking by mutual fund managers, Chevalier and Ellison (1997) build on the traditional risk-shifting insight and define risk-taking incentive as the sensitivity of a fund's value to its volatility. Our analysis in a dynamic setting revisits Chevalier and Ellison, and offers considerably different implications for managerial risk taking.[7] Within a dynamic asset allocation framework like ours, Carpenter (2000) examines the risk-taking behavior of a fund manager with a convex, option-based compensation. Such a payoff structure is not a subclass of the flow functions we consider since our manager always incurs a penalty (fund outflows) when her performance deteriorates, while a poor-performing manager in Carpenter enjoys a fixed safety net independent of her fund value. Due to this safety net feature, the manager's risk aversion considerations are suppressed in the underperformance region, implying unbounded risk exposure. This contrasts to our finding of an interior extremum at poor fund relative-performance, reflecting the tradeoff between the manager's risk-shifting and risk-aversion.

Building on the analysis of Carpenter and our paper, Hodder and Jackwerth (2004) analyze a hedge fund manager, incorporating further realistic features in the compensation structure along the lines of Goetzmann, Ingersoll, and Ross (2003). They also provide a thorough comparison of managerial risk taking arising in Carpenter, Goetzmann, Ingersoll, and Ross, and our analysis. Also related are Brennan (1993), Cuoco and Kaniel (2000), and Gomez and Zapatero (2003) who study equilibrium asset prices in an economy with agents compensated based on their performance relative to a benchmark, as well as the fund manager's career concerns problem studied by Arora and Ou-Yang (2000). Hugonnier and Kaniel (2002) endogenize (marketable) fund flows in a dynamic economy with a small investor and a noncompetitive fund manager. Under the derived flow function, however, the manager's optimal policy does not depend on her risk aversion, and does not entail any risk shifting. Berk and Green (2005) develop a rational model with a competitive capital market and decreasing returns to scale in active portfolio management. In the context of fund-value-maximizing managers, they deduce an empirically plausible fund flows-performance relationship even with no persistence in performance, but do not study risk-shifting behavior. Lynch

[7] A related area in corporate finance is the work on risk averse managers' risk-taking incentives induced by executive stock options (Lambert, Larcker, and Verrecchia (1991), Hall and Murphy (2000), Hall and Murphy (2002), Lewellen (2004)). Here, a risk-taking incentive is given by the sensitivity of the manager's certainty equivalent wealth to volatility. There are some similarities between the results obtained in this context and ours (see, especially Lewellen (2004)), however unlike in our model the manager is assumed to hold a pre-specified portfolio, and may affect risk exposure only through manipulating the company's stock price. The notion of implicit incentives was introduced by Fama (1980) and Holmstrom (1999), and applied to other related problems in corporate finance by, for example, Zwiebel (1995) and Huddart (1999).

and Musto (2003) suggest that the convexity in the flows-performance relationship is due to an abandonment option (replacement of personnel or technique after bad performance). Ross (2004) investigates the interaction between the manager's payoff and risk aversion within a general class of preferences and compensation structures, however, he focuses on fee schedules under which the objective function remains globally concave. More in the spirit of our analysis, in a dynamic portfolio choice framework, Cadenillas, Cvitanic, and Zapatero (2004) consider a principal-agent problem in which a risk-averse manager compensated with options chooses the riskiness of the projects she invests in.[8] There is also a recent literature examining asset allocation under benchmarking constraints. In a dynamic setting like ours, Tepla (2001), and Basak, Shapiro, and Tepla (2004) study the optimal policies of an agent subject to a benchmarking restriction.

The rest of the paper is organized as follows. Section 2 describes the model, solves for the optimal risk exposure of the manager under various fund-flow to relative-performance specifications, and computes the potential costs of active management to the investor due to managerial explicit and implicit incentives. Section 3 discusses the extension of our analysis to multiple sources of uncertainty and multiple stocks. Section 4 provides some empirical support for the implications of our theory. Section 5 concludes, and the Appendix provides the proofs and further details of our analysis.

2. Fund Manager's Implicit and Explicit Incentives

2.1 The Economic Setting

We adopt the familiar Black and Scholes (1973) economy for the financial investment opportunities. We consider a continuous-time, finite horizon, $[0, T]$ economy, in which uncertainty is driven by a Brownian motion w. Available for investing are a riskless money market account and a risky stock. The money market provides a constant interest rate r. The stock price, S, follows a geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dw_t,$$

where the stock mean return, μ, and volatility, σ, are constant.[9] Throughout, the notation σ^Z denotes the volatility (instantaneous standard deviation) of an Itô process Z satisfying $dZ_t/Z_t = \mu_t^Z dt + \sigma_t^Z dw_t$.

[8]There is a strand of literature, growing out of Bhatacharya and Pfleiderer (1985), investigating optimal contracting (or explicit incentive problems) in the context of delegated portfolio management, where the manager typically has superior information or ability, or expends costly effort. Also in this vein is Starks (1987). A recent example of such a modeling approach is Becker, Ferson, Myers, and Schill (1999), who like us also consider a manager concerned with her relative performance. In a similar sprit is Dybvig, Farnsworth, and Carpenter (2001), who also study restrictions on the investment opportunity set (trading strategies) as part of an optimal contract.

[9]In this Black and Scholes-type setting, the focus is on systematic risk. The extension to multiple sources of uncertainty and multiple stocks where systematic risk is not the only consideration is in Section 3.

We consider a fund manager who dynamically allocates the fund's assets, initially valued at W_0, between the risky stock and the money market. Her portfolio value process, W, follows

$$dW_t = [(1-\theta_t)r + \theta_t\mu]\, W_t dt + \theta_t \sigma W_t dw_t\,, \tag{1}$$

where θ denotes the fraction of the portfolio invested in the risky stock, or the *risk exposure.* Consistent with the leading practice, the manager's compensation, due at the horizon T, is proportional to the terminal value of assets under management. Tying of compensation to performance provides the manager with an *explicit incentive* to increase the final value of the portfolio W_T. Perhaps just as significant to the manager's choices are *implicit incentives* underlying the money management industry. There, implicit incentives come in the form of the well-documented fund-flows to relative-performance relationship (see e.g., Chevalier and Ellison (1997)). If the manager does well relative to some benchmark (e.g., the stock market), her assets under management multiply due to the inflow of new investors' money; if she does poorly, a part of assets under her management gets withdrawn. Our model of this relationship draws on the estimation by Chevalier and Ellison. The benchmark relative to which her performance is evaluated, hereafter the *benchmark*, Y, is a value-weighted portfolio with a fraction β invested in the stock market and $(1-\beta)$ in the money market, following[10]

$$dY_t = (1-\beta) r Y_t dt + \beta (Y_t/S_t) dS_t = [(1-\beta)r + \beta\mu] Y_t dt + \beta\sigma Y_t dw_t\,.$$

The (continuously compounded) returns on the manager's portfolio and on the benchmark over the period $[0,t]$ are denoted by $R_t^W = \ln \frac{W_t}{W_0}$ and $R_t^Y = \ln \frac{Y_t}{Y_0}$, respectively, where we normalize $Y_0 = W_0$, without loss of generality. At the terminal date, the manager receives fund flows at rate f_T, specified as follows:

$$f_T = \begin{cases} f_L & \text{if} \quad R_T^W - R_T^Y < \eta_L, \\ f_L + \psi(R_T^W - R_T^Y) & \text{if} \quad \eta_L \le R_T^W - R_T^Y < \eta_H, \\ f_H \equiv f_L + \psi(\eta_H - \eta_L) & \text{if} \quad R_T^W - R_T^Y \ge \eta_H, \end{cases} \tag{2}$$

with $f_L, \psi > 0$, $\eta_L \le \eta_H \in \mathbb{R}$. As estimated by Chevalier and Ellison (Figure 1), the flow-performance relationship is flat for managers who are well below the market. When the relative performance reaches about $\eta_L = -8\%$, the flow function displays a convex "kink" followed by an upward-sloping (approximately) linear segment. At about $\eta_H = 8\%$, the relationship again becomes flat.[11] Hereafter, we refer to this flow specification as a *collar-type*, as it resembles a familiar collar

[10] Alternatively, the benchmark Y can be interpreted as a peer group performance. The optimal policies we derive in this paper would then constitute the best response of an individual manager. One may in principle attempt to go further and solve for a Nash equilibrium in the game amongst managers evaluated on their performance relative to the peer group. This, however, would require a manager to have full knowledge of other managers' preferences, portfolio compositions, etc.

[11] Of course, our form of f_T is an overly simplified version of the specification revealed by Chevalier and Ellison's estimation. Here, for expositional clarity, we do not consider two other "kinks" corresponding to the regions of extremely good and extremely bad performance relative to the benchmark. This is because, as will become clear, our analysis is of a local type: focusing on one convex region at a time. Alternative empirical estimations (e.g., Sirri and Tufano (1998)) find that the flow-performance relationship is flat in the underperformance region, but becomes convex for overperforming funds. We consider this and other possible fund-flow specifications in Section 2.3.

or a bull spread of option pricing, with a *lower threshold* η_L and an *upper threshold* η_H. The flow rate f_T is understood in the proportion-of-portfolio terms; for example if $f_T > 1$, the manager gets an inflow, otherwise if $f_T < 1$, gets an outflow. The manager is guided by constant relative risk aversion (CRRA) preferences, defined over the overall value of assets under management at time T:

$$u(W_T f_T) = \frac{(W_T f_T)^{1-\gamma}}{1-\gamma}, \ \gamma > 0\,, \tag{3}$$

where f_T directly enters through the utility, and not through the budget constraint, because future (time-T) fund flows are nontradable. We note that this payoff is consistent with a linear fee structure, predominantly adopted by mutual fund companies (e.g., Das and Sundaram (2003), Elton, Gruber, and Blake (2003)).[12] It turns out that this simple way of modeling fund flows is able to capture most of the insights pertaining to risk taking incentives of a risk-averse manager that we attempt to highlight. In Section 2.3, we discuss how our results extend to other fund-flows to relative-performance relationships, and also reinterpret the resulting payoff function of the manager as a compensation contract.

Absent implicit incentive considerations, the manager's optimal risk exposure, θ^N, henceforth the *normal risk exposure*, is given by (Merton (1971)):

$$\theta_t^N = \frac{1}{\gamma}\frac{\mu - r}{\sigma^2}.$$

By analogy, we define the risk exposure of the benchmark portfolio, θ^Y, as the fraction of the benchmark invested in the risky asset:

$$\theta_t^Y = \beta.$$

2.2 Manager's Risk Taking Incentives

The optimization problem of the manager is given by:

$$\max_{\theta,\, W_T} E[u(W_T f_T)] \tag{4}$$

$$\text{subject to} \qquad dW_t = [r + \theta_t(\mu - r)]\, W_t dt + \theta_t \sigma W_t dw_t\,,$$

with W_0 given and f_T as defined in (2). This problem is nonstandard in that it is nonconcave over a range of W_T, where the range is dependent on the performance of the stochastic benchmark Y, and where the implicit incentives due to fund flows introduce a local convexity. The empirical literature on fund-flows to relative-performance relationship clearly indicates that convexities are inherent in the mutual fund managers' problems. A convexity in the manager's objective gives rise to a range

[12] In particular, the fund company may have a linear fee structure, $\alpha W_T f_T$, $\alpha > 0$. Such a linear structure would be optimal in our model absent implicit and explicit incentives, i.e., if $f_T = 1$ and a hypothetical investor had CRRA preferences with γ. However, the presence of incentives leads to a considerable cost to investors, as demonstrated in Section 2.4. Our specification does not capture the case of fulcrum fees, which are less common, but the model can be extended to incorporate them.

of terminal portfolio values W_T over which the manager is risk-loving (and the objective lies below its concavified version). In our model such a range, which we refer to it as the *risk-shifting range*, is finite. Interestingly, in contrast to Ross (2004), the shape of the manager's objective function in this range is immaterial for the risk-taking choices of the manager. Only in the region in which the objective is strictly concave (the case considered in Ross), does the shape of the objective start playing a role and the Ross decomposition applies. For the collar-type fund flow specification, the occurrence of the risk-shifting range is due to a convex kink at the lower threshold η_L. The (upward-sloping) intermediate part of the flow function, triggered by the manager's exceeding η_L, may or may not be fully subsumed within the risk-shifting range (see Appendix B for an elaboration). The former case occurs under the following condition:

Condition 1. $\frac{\gamma}{1-\gamma}\left(\frac{f_H+\psi}{f_L}\right)^{1-1/\gamma}+\frac{f_H+\psi}{f_H}-\frac{1}{1-\gamma}\geq 0\,,$

where, recall, $\psi \equiv (f_H - f_L)/(\eta_H - \eta_L)$. This is satisfied for empirically plausible parameters of the flow-performance relationship ($f_L = 0.8$, $f_H = 1.5$, $\beta = 1.0$, $\eta_L = -0.08$, $\eta_H = 0.08$), calibrated from Chevalier and Ellison's Figure 1, and a range of managerial risk aversion $\gamma \in (0,\, 1.56)$. More generally, for low enough risk aversion γ or threshold differential $|\eta_H - \eta_L|$, Condition 1 is always satisfied. This is because the manager would like to "gamble" in the risk-shifting range, and with lower risk aversion or threshold differential, the incentives to gamble are more pronounced, thereby enlarging the risk-shifting range so as to fully subsume the intermediate part of the flow function f_T. We first focus on the case where Condition 1 holds and an explicit characterization for the optimal policy obtains; the other case is considered in Section 2.3.

As is well known (e.g., Karatzas and Shreve (1998)), the driving economic state variable in an agent's dynamic investment problem is the so-called state price density. In the complete-markets Black and Scholes (1973) economy, this state price density process, ξ, is given by $d\xi_t = -r\xi_t dt - \kappa\xi_t dw_t$, where $\kappa \equiv (\mu - r)/\sigma$ is the constant market price of risk in the economy. Proposition 1 characterizes the solution to (4) in terms of the state variable ξ.[13]

Proposition 1. *Under Condition 1, the optimal risk exposure and terminal wealth of a fund manager facing implicit incentives are given by*

[13] Our proof of Proposition 1 exploits the dependence of the benchmark Y on the economic state variable ξ. There is an alternative to this direct method, which is presented in the proof of Proposition 2 dealing with multiple sources of uncertainty and multiple stocks.

(a) for economies with $\theta^N > \theta^Y$, letting $\xi_a > \hat{\xi}$ satisfy $g(\xi_a) = 0$, we have

$$\begin{aligned}
\hat{\theta}_t &= \theta^N + \Big[\mathcal{N}(d(\hat{\kappa},\xi_a)) - \mathcal{N}(d(\hat{\kappa},\hat{\xi}))\Big]\,(\gamma/\hat{\kappa}-1)\,A\,\theta^N Z(\hat{\kappa})\xi_t^{-1/\hat{\kappa}}/\hat{W}_t \\
&\quad + \Big\{\Big[\phi(d(\hat{\kappa},\xi_a)) - \phi(d(\hat{\kappa},\hat{\xi}))\Big]\,A\,Z(\hat{\kappa})\xi_t^{-1/\hat{\kappa}} \\
&\quad + \Big[\phi(d(\gamma,\hat{\xi}))f_H^{(1/\gamma-1)} - \phi(d(\gamma,\xi_a))f_L^{(1/\gamma-1)}\Big]\,Z(\gamma)(\hat{y}\xi_t)^{-1/\gamma}\Big\}\frac{\gamma\theta^N}{\kappa\sqrt{T-t}\,\hat{W}_t}\,, \\
\hat{W}_T &= \tfrac{1}{f_H}J\left(\tfrac{\hat{y}}{f_H}\xi_T\right)1_{\{\xi_T<\hat{\xi}\}} + e^{\eta_H}Y_T 1_{\{\hat{\xi}\le\xi_T<\xi_a\}} + \tfrac{1}{f_L}J\left(\tfrac{\hat{y}}{f_L}\xi_T\right)1_{\{\xi_a\le\xi_T\}}
\end{aligned}$$

(b) for economies with $\theta^N < \theta^Y$, letting $\xi_b < \hat{\xi}$ satisfy $g(\xi_b) = 0$, we have

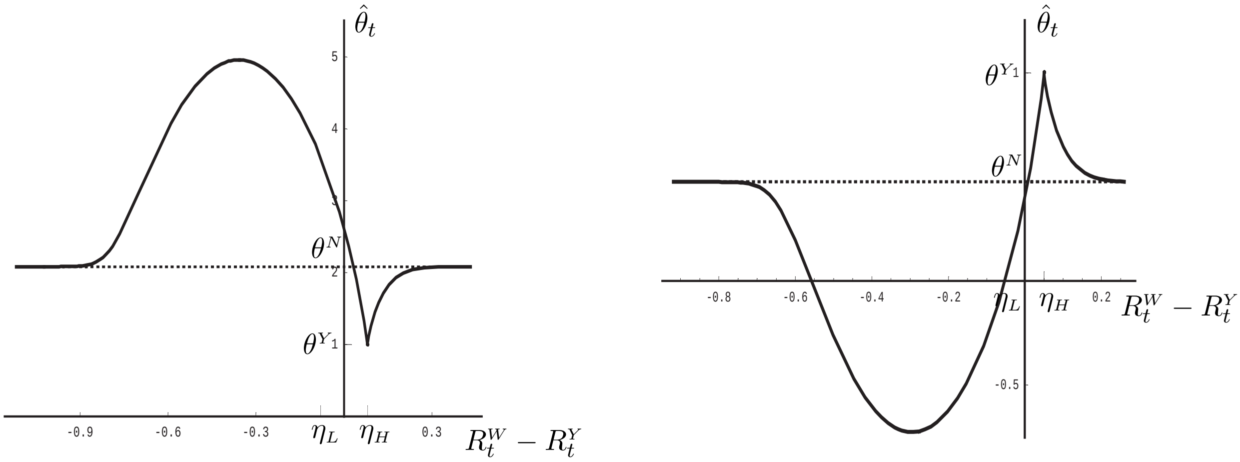
$$\begin{aligned}
\hat{\theta}_t &= \theta^N + \Big[\mathcal{N}(d(\hat{\kappa},\hat{\xi})) - \mathcal{N}(d(\hat{\kappa},\xi_b))\Big]\,(\gamma/\hat{\kappa}-1)A\,\theta^N Z(\hat{\kappa})\xi_t^{-1/\hat{\kappa}}/\hat{W}_t \\
&\quad + \Big\{\Big[\phi(d(\hat{\kappa},\hat{\xi})) - \phi(d(\hat{\kappa},\xi_b))\Big]\,A\,Z(\hat{\kappa})\xi_t^{-1/\hat{\kappa}} \\
&\quad + \Big[\phi(d(\gamma,\xi_b))f_L^{(1/\gamma-1)} - \phi(d(\gamma,\hat{\xi}))f_H^{(1/\gamma-1)}\Big]\,Z(\gamma)(\hat{y}\xi_t)^{-1/\gamma}\Big\}\frac{\gamma\theta^N}{\kappa\sqrt{T-t}\,\hat{W}_t}\,, \\
\hat{W}_T &= \tfrac{1}{f_L}J\left(\tfrac{\hat{y}}{f_L}\xi_T\right)1_{\{\xi_T<\xi_b\}} + e^{\eta_H}Y_T 1_{\{\xi_b\le\xi_T<\hat{\xi}\}} + \tfrac{1}{f_H}J\left(\tfrac{\hat{y}}{f_H}\xi_T\right)1_{\{\hat{\xi}\le\xi_T\}},
\end{aligned}$$

where in all economies $\hat{y}$ solves $E[\xi_T\hat{W}_T] = W_0$, $J(\cdot)$ is the inverse function of $u'(\cdot)$, $\mathcal{N}(\cdot)$ and $\phi(\cdot)$ the standard-normal cumulative distribution and density functions respectively, $\hat{\kappa} = \kappa/(\beta\sigma)$, $\hat{\xi} = (\hat{y}A^\gamma/f_H^{1-\gamma})^{1/(\gamma/\hat{\kappa}-1)}$, $A = W_0 e^{\left[\eta_H/T+(1-\beta)r+\beta\left(\mu-\beta\sigma^2/2-(r+\kappa^2/2)\sigma/\kappa\right)\right]T}$, $Z(v) = e^{\frac{1-v}{v}\left(r+\frac{\kappa^2}{2v}\right)(T-t)}$, $g(\xi) = \left(\gamma\left(\frac{\hat{y}}{f_L}\xi\right)^{1-1/\gamma} - \left(\frac{\xi^{1/\hat{\kappa}}}{Af_H}\right)^{\gamma-1}\right)/(1-\gamma) + \hat{y}A\xi^{1-1/\hat{\kappa}}$, $d(v,x) = \left(\ln\frac{x}{\xi_t} + \left(r+\frac{2-v}{2v}\kappa^2\right)(T-t)\right)/(\kappa\sqrt{T-t})$, and $\hat{W}_t$ is as given in Appendix A. Economies with $\theta^N = \theta^Y$ are described in Appendix A.

Proposition 1 reveals that the manager's optimal behavior has a different pattern depending on whether the benchmark is riskier than her normal policy (economies (a)) or not (economies (b)). We note that both types of economies, (a) and (b), are empirically plausible since each economy is identified by conditions involving managerial risk aversion γ, which need not equal that of a representative agent. The implications for optimal risk taking are best highlighted by plotting the manager's state-dependent risk exposure as a function of her performance relative to the benchmark, as presented in Figure 1.

There are two considerations affecting the manager's behavior. First is her attitude towards risk, driving the normal policy, second is the risk-shifting incentive induced by the kink in the manager's payoff when her relative performance is at the lower threshold η_L. As emphasized in the vast risk-shifting literature (originating from Jensen and Meckling (1976)), to increase her portfolio value, the manager has an incentive to distort her normal policy by boosting her portfolio volatility. Indeed, the manager's risk-loving behavior over a range of terminal payoffs gives rise to a hump in her optimal risk exposure, as clearly pronounced in both panels of Figure 1. In panel (a), this hump reflects an optimal choice to increase her portfolio volatility over the risk-shifting range.

In panel (b), in contrast, the volatility is reduced. The latter direction is somewhat unexpected: how can "gambling" to finish ahead of the benchmark be consistent with a decrease in portfolio volatility? Simply, in the context of relative performance evaluation, any strategy entailing a deviation from the benchmark is inherently risky. By taking on more systematic risk than that of the benchmark (boosting portfolio volatility), the manager gambles to improve her relative standing when the benchmark goes up. Similarly, by taking on less systematic risk than that of the benchmark (reducing volatility), the manager bets on improving her relative performance when the benchmark falls. The direction of the manager's choice of a deviation from the benchmark depends on her risk aversion: a more risk tolerant manager decides to boost volatility, while, in utility terms, it is cheaper for a relatively risk averse manager to do the opposite, despite the positive risk premium offered by the risky asset. We note that this implication is very general, and holds for all alternative flow-performance specifications considered in Section 2.3. This suggests that volatility is not the most accurate measure of risk under relative performance evaluation. As follows directly from our closed-form expressions, the quantity measuring risk taking in our model is the distance between the volatility of the manager's portfolio and that of the benchmark $|\sigma_t^W - \sigma_t^Y| = \sigma|\theta_t - \theta_t^Y|$.[14] In both panels of Figure 1, $|\sigma_t^W - \sigma_t^Y|$ increases sharply over the risk-shifting range, while the manager's portfolio volatility σ_t^W may or may not increase.



(a) Economies with $\theta^N > \theta^Y$.

(b) Economies with $\theta^N < \theta^Y$.

Figure 1. The manager's optimal risk exposure. The solid plots are for the optimal risk exposure, and the dotted plots are for the manager's normal risk exposure. In economies (a) parameter values are $\gamma = 1.5$, $f_L = 0.8$, $f_H = 1.5$, $\beta = 1.0$, $\eta_L = -0.08$, $\eta_H = 0.08$, $\mu = 0.1$, $r = 0.02$, $\sigma = 0.16$, $W_0 = 1$, $t = 0.75$, $T = 1$; in economies (b) $\gamma = 2$, $f_L = 0.8$, $f_H = 1.5$, $\beta = 1.0$, $\eta_L = -0.08$, $\eta_H = 0.08$, $\mu = 0.1$, $r = 0.02$, $\sigma = 0.29$, $W_0 = 1$, $t = 0.75$, $T = 1$.[15]

[14]Indeed, in the proof of Proposition 2, we provide a formal justification for focusing on this distance, where we demonstrate that the relevant state variable in the optimization problem of the manager (4) is her *relative* portfolio value W/Y, whose volatility is given by $|\sigma_t^W - \sigma_t^Y|$. We also note that in this Section, the only type of gambles the manager is allowed to take on are systematic gambles. A natural question is whether the optimal risk-taking behavior we derive here would persist if the manager were allowed access to a market for idiosyncratic gambles. We examine this possibility in Section 3.

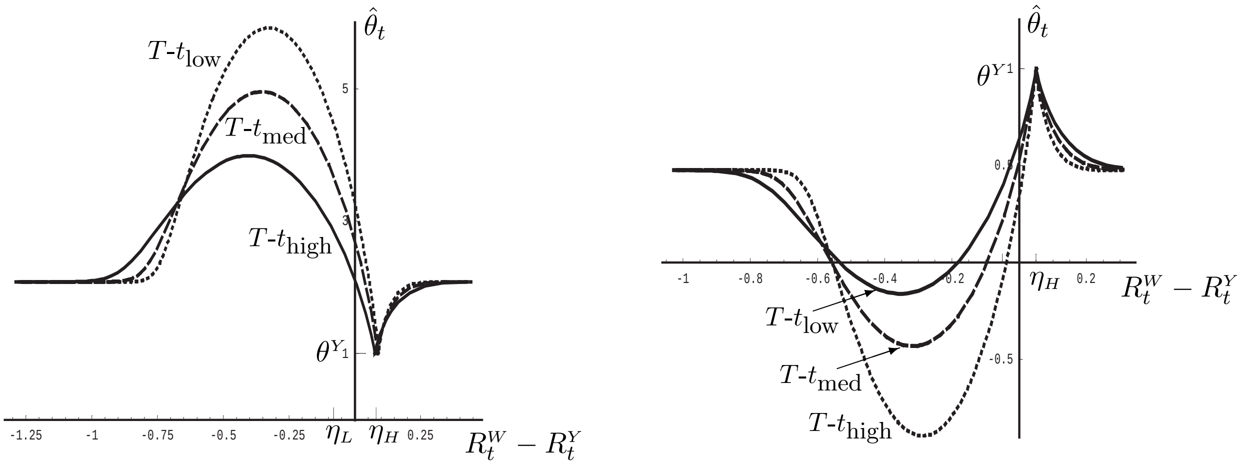
[15]The figure is typical. The parameter values of the flow function are calibrated according to the estimation in

Furthermore, when the manager is considerably behind or ahead of her benchmark, her flows-induced implicit incentives are weak, and so her optimal policy converges to her normal policy. Consequently, the optimal risk-shifting range is finite, positioned in the neighborhood of the convex kink in the flow-performance relationship. Over that range, the optimal risk exposure is certainly not infinite, reflecting a tradeoff between the managerial risk shifting and risk aversion. The maximum risk taking obtains somewhere inside this range; in Figure 1, the maximum corresponds to a point deep in the underperformance region, well below the lower threshold η_L. The minimum is around the upper threshold η_H, reflecting the positioning of the upper boundary of the risk-shifting range (Appendix B). Here, it would be useful to contrast our results on the manager's optimal risk taking to measures of risk-taking incentives as defined in the corporate finance literature, typically under the assumption of agents' risk neutrality. For example, Green and Talmor (1986), in the context of the asset substitution problem, define the risk-taking incentive as the sensitivity of the value of the equityholders' option-like payoff to "changes in investment risk" (variability of the underlying cash flow). In option pricing, this measure is referred to as vega, the partial derivative of an option's (portfolio) value with respect to the underlying volatility. The risk-taking incentive, as defined in corporate finance, then captures the strength of the (value-maximizing) manager's desire to increase her portfolio volatility relative to some state-independent status quo asset allocation. This risk taking incentive is strongest when vega of the manager's payoff achieves its maximum. Adopting this measure for the relative-performance-based payoff, Chevalier and Ellison (1997) argue that the "incentive to increase or decrease risk will always be maximized at the point at which the flow-performance relationship has a kink and will decline smoothly to zero at extreme performance levels." As evident from our Figure 1, our measure of risk taking, the optimal risk exposure, conflicts with this prediction $-$ in fact, in panel (b) the risk exposure is around zero at the convex kink η_L. Intuitively, instead of merely taking a partial derivative of the manager's value function with respect to volatility, we take this partial derivative and equate it to zero to derive the optimal volatility for each level of relative performance.[16]

The described wave-shape optimal policy finances the manager's terminal portfolio value which displays three distinct patterns depending on the state of the world. In the extreme states (low ξ_T or high ξ_T), the manager behaves as if the fund flows were constant at the low f_L or high f_H rate. In addition, there is an extended intermediate region in which the manager mimics the benchmark, "locking in" her gains by holding the benchmark when her performance reaches the upper threshold η_H. The nonconcavity of the manager's problem gives rise to a discontinuity in the optimal wealth profile at $\xi_T = \hat{\xi}$, responsible for the risk-shifting range.

Chevalier and Ellison.

[16]Note that our endogenous wave-shape pattern of risk exposure does not converge to the corporate-finance (bell-shaped) measure of risk-taking incentives even as the risk aversion coefficient of the manager tends to zero. This is because the proper limit of the preferences of our manager is a linear function over the range of positive values of terminal wealth, coupled with a restriction that wealth cannot fall below zero (the negativity of wealth is ruled out by the Inada conditions). This function is (weakly) concave, so we do not get a risk-neutral (linear) objective even in the limit.



(a) Economies with $\theta^N > \theta^Y$. (b) Economies with $\theta^N < \theta^Y$.

Figure 2. Dynamics of the manager's optimal risk exposure. T-$t_{\text{high}} = 0.5$, T-$t_{\text{med}} = 0.25$, and T-$t_{\text{low}} = 0.15$. The remaining parameter values are the same as in Figure 1.

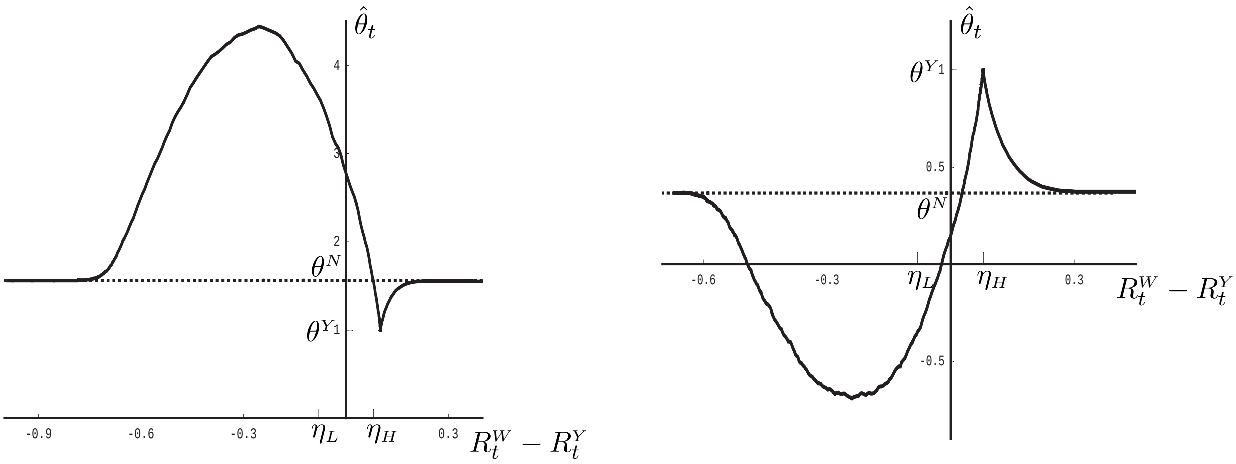
The manager's optimal trading strategy throughout the year reflects the anticipation of the year-end drive to avoid the suboptimal range of the terminal portfolio returns. As evident from Figure 2, she does not wait till the year-end to see how her returns play out, to then take a gamble right before the terminal date if necessary. Rather, she starts tilting her risk exposure around the suboptimal range well in advance, displaying a hump in the risk exposure as in Figure 1. However, the more opportunities she has to adjust her portfolio in the future, the less risk exposure she is willing to bear today. The risk aversion (normal policy) considerations dominate early in the year, substantially tempering the risk-shifting considerations and bringing the optimal policy closer to normal, but as time progresses, the risk-shifting motive grows stronger, and hence the magnitude of risk taking around the suboptimal range of portfolio returns grows. Additionally, the range over which the risk exposure displays a risk-shifting-induced hump is wider early in the year, and shrinks monotonically as the horizon approaches, with the minimum risk taking point converging to η_H. This latter pattern is clearly pronounced in Figure 2b; it is, however, less clear in Figure 2a, where for reasonable parameter values, the difference between the point at which the manager minimizes her deviations from the benchmark and the upper threshold η_H is positive but very small, not distinguishable to the eye.

2.3 Alternative Flow-Performance Specifications and Applications

This Section considers generalizations of our baseline analysis to alternative specifications of the fund-flow to relative-performance relationship and also attempts to reinterpret the ensuing manager's payoff functions as compensations contracts.

A. Collar-Type (continued)

We first consider the collar-type fund-flow specification (2) for which Condition 1 is violated. In this case, as further elaborated in Appendix B, the shape of the intermediate region of the fund flow function starts playing a role in determining the manager's optimal policy. An analytical solution is not available for this case because the manager's optimal terminal portfolio value displays four regions of distinct behavior, and so we solve for the optimal policy numerically (see Appendix B for an elaboration on the form of the solution and Appendix C for the details of our numerical procedure). The solution is presented in Figure 3.



(a) Economies with $\theta^N > \theta^Y$.

(b) Economies with $\theta^N < \theta^Y$.

Figure 3. The manager's optimal risk exposure for the case where Condition 1 is not satisfied. The solid plots are for the optimal risk exposure, and the dotted plots are for the manager's normal risk exposure. In panel (a), $\eta_H = 0.10$, and in both panels $\mu = 0.08$. The remaining parameter values are as in Figure 1.

We again find that the optimal risk exposure of the manager displays two different patterns, depending on whether the economies are of type (a) or (b). In the latter case, the manager optimally decreases the volatility of her portfolio. Overall, despite the more complex nature of the solution, the manager's optimal behavior displayed in Figure 3 is quite similar to that derived in Section 2.2. In other words, although theoretically the nature of the solution affects the positioning of the maximum and especially the minimum of optimal risk exposure (which is distinctly away from η_H), these differences are too small to be visibly reflected in Figure 3.

It is useful to draw a parallel between the collar-type flow function and executive compensation contracts in the US. As documented by Murphy (1999), the most prevalent form of annual bonuses offered to executives is the so-called 80/120 plan. Under an 80/120 plan, the manager takes home only the base salary if her performance does not exceed 80% of a so-called performance standard, set in advance. The point 80% below the performance standard is known as the "performance threshold," at which the manager's compensation jumps up, and then increases continuously over an "incentive zone". The shape of the incentive zone is typically linear, although Murphy reports some instances in which the shape is concave or convex. Annual bonuses are capped once the performance exceeds 120% of the performance standard. (See Section 2.3 of Murphy, and particularly Figure 5, for further details.) Under the plausible assumption of the base salary and hence the bonus (typically a percentage of base) being proportional to company's size, adopting a performance standard based on performance relative to a benchmark, we can attempt to reinterpret our manager's payoff function as an executive's base salary plus an annual bonus. Our collar-type specification is indeed very similar to an 80/120 plan; the only feature that is missing is the discontinuity at the performance threshold, which has been argued to affect a manager's incentives in a significant way (Murphy, p. 2507). Since the literature on compensation typically looks at sensitivity-based measures of risk-taking incentives, our measure would potentially yield rather different conclusions pertaining to risk taking, as highlighted in Section 2.2. Moreover, as we demonstrate in Proposition 1, for the case where Condition 1 holds, the shape of the incentive zone—concave, convex or any other—is simply not relevant for the manager's optimal choice, potentially defeating the intended purpose of the incentive zone.

B. Digital

As highlighted in the discussion above, a realistic model of annual bonuses requires a discontinuity in the manager's payoff at a performance threshold. Here, we isolate the effects of such a jump in the manager's payoff by specializing the fund flow function f_T to have the following simple form:

$$f_T = \begin{cases} f_L & \text{if} \quad R_T^W - R_T^Y < \eta, \\ f_H & \text{if} \quad R_T^W - R_T^Y \geq \eta, \quad 0 < f_L \leq f_H,\ \eta \in \mathbb{R}\,. \end{cases}$$

If her year-end relative performance is above the threshold η, she gets a high rate f_H, otherwise a low rate f_L. For this form of the manager's payoff, we can again obtain an analytical characterization of her optimal risk exposure, which is given by the same expression as in Proposition 1 but with η_H now replaced by the performance threshold η.

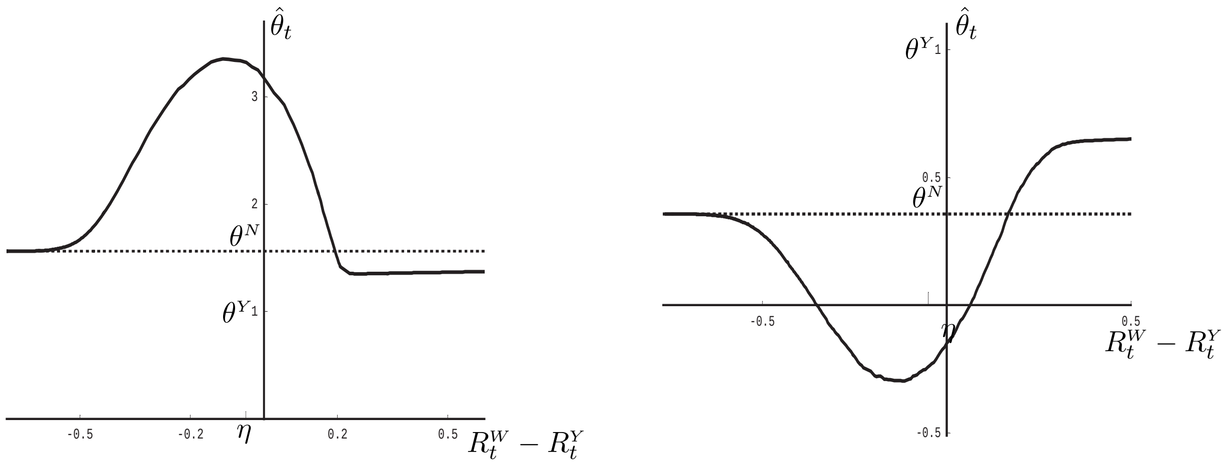
It is worth commenting on the risk taking incentives induced by the jump in the fund flows relationship. We recall from Figures 1–2 that the risk-taking incentives are minimized around the performance threshold η. This is somewhat surprising in light of the literature arguing that a discontinuity induces a peak in risk-taking since the slope at the performance threshold is infinite (see e.g., Murphy (1999)). This inconsistency in the implications is again due to the differences in sensitivity- versus optimality-based measures such as ours. When a manager is risk averse, she strives to take on as little risk (as small a gamble) as necessary to exceed the performance threshold. Thus, a small increase in the risk exposure is sufficient to achieve this when the performance is right below the threshold, where the payoff's sensitivity to volatility is extremely high. As the manager falls further behind, she needs a bigger gamble to catch up with the benchmark, and so her risk taking keeps increasing in the underperformance region until it reaches a maximum where the risk shifting incentives are counterbalanced by her risk aversion.

C. Linear-Convex and Other Specifications

We now consider a fund-flows to relative-performance relation of the type documented by Sirri and Tufano (1998), where the flows become increasingly sensitive to performance in the region of good performance:

$$f_T = \begin{cases} f_L & \text{if} \quad R_T^W - R_T^Y < \eta, \\ f_L + \psi\left(e^{(R_T^W - R_T^Y)} - e^{\eta}\right) & \text{if} \quad R_T^W - R_T^Y \geq \eta, \end{cases}$$

with f_L, $\psi > 0$ and $\eta \in \mathbb{R}$.[17] Under this linear-convex specification, the flows are not sensitive to bad performance, however, they are very sensitive to good performance (increase exponentially upon exceeding the threshold η). We solve for the manager's optimal behavior numerically and present the solution in Figure 3.



(a) Economies with $\theta^N > \theta^Y$. (b) Economies with $\theta^N < \theta^Y$.

Figure 4. The manager's optimal risk exposure under a linear-convex flow-performance specification. The solid plots are for the optimal risk exposure, and the dotted plots are for the manager's normal risk exposure. The parameter values $f_L = 0.97$, $\psi = 1.6$, $\eta = -0.05$ are calibrated to match the estimated relation in Sirri and Tufano (1998). The remaining parameter values are the same as in Figure 3.

We again note the familiar pattern of risk exposure: a hump over the risk-shifting range and two types of deviations from the benchmark, distinguishing panels (a) and (b). The intuition for these patterns is as in our baseline specification of Sections 2.1–2.2. The main difference here occurs in

[17]For brevity, we omit a technical condition, which amounts to imposing a lower bound on η, guaranteeing that the manager's objective function is strictly concave in the overperformance region. This condition is easily satisfied under our parameterization.

the overperformance region, where the flows are always convex. When considerably ahead of the benchmark, the manager's optimal policy no longer tends to her normal policy. First, a convex flow function combined with a concave objective alters the effective risk aversion of the manager, which now tends to $2\gamma - 1$ (since in the limit her payoff is quadratic in portfolio value W). Second, even in the limit, the manager's payoff depends on the level of the benchmark (a feature absent in our previous specifications), and hence the manager holds an additional hedge portfolio (Appendix C). Due to the insensitivity of flows to bad performance, however, the limit of the risk exposure in the underperformance region coincides with the manager's normal policy, as before.

We have also considered a linear-linear specification of the flow-performance relationship, where the second segment has a higher slope, hence resembling a call option. Such a specification is becoming increasingly relevant since it can be interpreted as an asymmetric fee structure of mutual fund managers. Asymmetric fees are not common in the US mutual fund industry, but are becoming more prevalent amongst the European mutual funds (about 12% of 4000 funds according to Fitzrovia, a Lipper company). The resulting optimal risk exposure is similar to that presented in Figure 4, with the only difference being that the manager again behaves as if her risk aversion were γ in the limit of good performance. We omit the figure here for brevity.

Finally, it is of interest to conjecture what a resulting risk exposure may look like for a general payoff structure $W_T f_T$. We believe that the bulk of our results holds locally for every region in which $u(W_T f_T)$ is nonconcave. If such a region includes $W_T = 0$ or $W_T = \infty$, then at the global maximum (or minimum), the manager's risk exposure can be infinite (a corner solution) or not well-defined. Otherwise, the manager's risk exposure is bounded from above and below for each t, and the emerging wave-shape pattern of risk taking incentives is along the lines of that described in the preceding analysis.

We note that our analysis and insights may possibly be applicable to address other issues in finance, where nonconcavities are inherent in the payoff structure and where agents can be argued to be (effectively) risk averse. We have alluded to some reinterpretations of our flow function as bonuses of executives. Some other natural avenues include modeling stock-option-based executive compensation and option-like compensation of hedge fund managers. Nonconcave payoffs also arise in other applications. For example, in banking "gambling for resurrection" arises due to nonconcavities introduced by deposit insurance; in corporate finance, the classical asset substitution problem is due to shareholders having a nonconcave payoff. Since in most of these applications nonconcavities in the payoff do not arise over a stochastic range as in our problem with a stochastic relative performance benchmark, it is worth pointing that a riskless benchmark is a special case of our analysis. The pattern of optimal risk exposure will resemble that in our Figure 1a. An additional application that fits our digital payoff specification is an election, with the two possibilities representing the outcomes of being elected or not. Our optimal risk-shifting range would then represent the behavior of a risk averse candidate who is behind in the polls. Similar incentives may be relevant in the case of financial analysts or sports experts, leading to more extreme predictions when they are behind their peers.

Remark 1. ("Safety nets" and relation to Carpenter (2000)) We here discuss why our manager's behavior we uncover is at odds with the insights drawn from Carpenter. Carpenter studies the risk taking of a risk-averse manager paid with a call option on the assets she controls. As Carpenter highlights, such a convex compensation structure could potentially arise from the documented fund flow-performance relationship. We may attempt to interpret Carpenter's model as reduced form for such an implicit incentive. The closest way to obtain Carpenter's "safety net" at poor fund performance within our model, is to consider a two-region fund flow function with $f_L = K/W_T$ and f_H constant, guaranteeing the floor K for portfolio values below K, and a payoff linear in W_T for levels above K. The benchmark Y is nonstochastic ($\theta^Y = 0$) for most of Carpenter's analysis. Note that this form of fund-flows to relative-performance relationship entails a counterfactual implication that fund flows increase as the fund value decreases when performing poorly (for low W_T, $f_L > f_H$), and tends to (positive) infinity at zero fund value. This behavior highlights the incentive effects of a "safety net", along the lines of the risk-shifting story of Jensen and Meckling (1976). Our manager, who is not rewarded for poor performance and is instead penalized by low fund flow for low W_T, behaves considerably differently. The risk aversion considerations play an important role in the underperformance region in counteracting risk-shifting, thus ruling out unbounded risk exposure. This feature will also be present in a model where a manager can get fired when the fund return is low, since the manager would optimally choose to limit the size of the gamble she takes to avoid the states in which she gets fired. Another finding of our analysis is uncovering of economies (b). Although a stochastic benchmark is considered in Carpenter, her model is able to uncover only economies of type (a).

2.4 Costs of Active Management

In this Section, we assess the economic significance of the manager's adverse behavior. Towards this, we consider an investment policy associated with the manager acting in the best interest of fund investors, fully ignoring her explicit and implicit incentives. A hypothetical fund investor is assumed to have CRRA preferences, $u_I(W_T) = \frac{W_T^{1-\gamma_I}}{1-\gamma_I}$, $\gamma_I > 0$, over the horizon wealth W_T. The investor is passive in that he delegates all his initial wealth, W_0, to the manager to invest. The decision to delegate, exogenous here, captures in a reduced form the choice to abstain from active investing due to various imperfections associated with money management (participation and information costs, time required to implement a dynamic trading strategy, transaction costs, behavior limitations), or simply because he believes that the manager has better information or ability. The manager's investment policy that maximizes the investor's utility is given by $\theta^I = \frac{1}{\gamma_I}\frac{\mu - r}{\sigma^2}$.

In order to evaluate the economic significance of the manager's incentives, we compute the utility loss to the investor of the manager's deviating from the policy θ^I.[18] Following Cole and Obstfeld (1991), we define a cost-benefit measure, $\hat{\lambda}$, reflecting the investor's gain/loss quantified in units of his initial wealth:

$$V^I((1+\hat{\lambda})W_0) = \hat{V}(W_0)\,,$$

where $V^I(\cdot)$ denotes the investor's indirect utility under the policy absent incentives θ^I, and $\hat{V}(\cdot)$ his indirect utility under the optimal policy accounting for incentives $\hat{\theta}$. In order to disentangle the implications of explicit and implicit incentives of the manager, we decompose the total cost-benefit measure into two components: λ^N and λ^Y. The former captures the effects of the manager's attitude towards risk driving her normal policy, while the latter the effects of implicit incentives. In particular, λ^N solves $V^I((1+\lambda^N)W_0) = \hat{V}(W_0;\, f_T = 1)$, where $\hat{V}(W_0;\, f_T{=}1)$ denotes the investor's indirect utility absent implicit incentives only, and λ^Y solves $1+\hat{\lambda} = (1+\lambda^N)(1+\lambda^Y)$.

The main parameter governing the gain/loss due to explicit incentives is the manager's risk aversion, γ. Absent implicit incentives, the further γ deviates from the investor's risk aversion, γ_I, the larger the discrepancy between the optimal risk exposure of the manager, θ^N, and that optimal for the investor, θ^I, and consequently the higher the loss to the investor. As reported in Tables 1a and 1b, the loss due to explicit incentives, λ^N, is zero when the manager and the investor have the same attitude towards risk, $\gamma = \gamma_I(=2)$. However, for $\gamma = 0.5$ such a loss can be quite significant: 28.86% in economies (a) and 8.13% in economies (b).[19]

The strength of the implicit incentives is dependent upon the risk-shifting behavior induced by fund flows. Absent explicit incentives, the more the manager engages in gambling behavior when underperforming, the more she deviates from the investor's desired risk exposure. Table 1 reveals that the loss to the investor due to implicit incentives, λ^Y, increases with (i) the implicit reward for outperformance, $f_H - f_L$, (ii) typically the riskiness of the benchmark, θ^Y, and (iii) the flow threshold differential, $\eta_H - \eta_L$. For the largest implicit reward considered in Table 1, $f_H - f_L = 1.1$, the loss is 10.26% in economies (a) and 6.88% in economies (b). Additionally, the effects of implicit incentives are most pronounced for relatively risky benchmarks. For example, Table 1a reports the cost due to implicit incentives to be 11.79% for the riskiest benchmark we consider ($\theta^Y = 1.5$), and in Table 1b a corresponding cost of 8.45%. The explicit and implicit incentives effects reinforce each other in harming of the investor. Table 1 reports the total cost due to both explicit and implicit

[18] This measure is, of course, subject to a caveat that investors *have chosen* active management. This measure does not take into account the potential costs to investors associated with implementing their optimal strategy themselves.

[19] The values reported in Table 1 are for the model parameters, calibrated to conform with the observed market dynamics and capturing the observed flow-performance relationship for mutual funds. The reported range of managerial risk aversion coefficient γ is smaller in Table 1a than in Table 1b because the condition for economies (a) to occur imposes an upper bound on γ, equaling 2.34 for our baseline calibration. The market parameters in economies (b) represent "unfavorable" market conditions designed to temper the manager's normal risk exposure below that of the benchmark assumed to be the stock market. Although we do not frequently observe mutual fund managers holding a leveraged portfolio, the standard argument (Merton (1971) applied to parameter estimates based on historical data) predicts that they should. This observation is related to the discussion whether very high historical equity premium can be reconciled with a typical agent's preferences initiated by Mehra and Prescott (1985).

incentives, $\hat{\lambda}$ ranging from 2.27% to 48.16%.

3. Multiple Sources of Risk and Multiple Stocks

Until now, we have adopted the Black and Scholes (1973) specification for the financial investment opportunities featuring one riskless and one risky asset. Consequently, the decision of the fund manager has been the allocation of assets between the risky and riskless securities. This setting has served as the simplest possible setting, which allowed us to highlight the most important insights pertaining to risk taking incentives of the fund manager. In real life, however, the decision of the manager often involves allocating her portfolio between different stocks, rather than between stocks versus bonds. Moreover, unlike in our baseline model, managers may wish to adjust their portfolio riskiness through taking on idiosyncratic rather than systematic risk. Thus, it is of interest to examine a setup in which one can make a distinction between the effects of systematic and idiosyncratic risks on the manager's decisions.

Towards this end, we extend our model of Section 2.1 to multiple sources of uncertainty and multiple stocks. Each stock price, S_i, follows

$$dS_{it} = \mu_i S_{it} dt + \sigma_i S_{it} dw_t \,, \qquad i = 1, \ldots, n,$$

where the stock mean returns $\mu \equiv (\mu_1, \ldots, \mu_n)^\top$ and the nondegenerate volatility matrix $\sigma \equiv \{\sigma_{ij},\ i,\, j = 1, \ldots n\}$ are constant, and $w = (w_1, \ldots, w_n)^\top$ is an n-dimensional standard Brownian motion. The benchmark relative to which the manager is evaluated, Y, is now a value-weighted portfolio with fractions $\beta \equiv (\beta_1, \ldots, \beta_n)^\top$ invested in the stocks and 1-$\beta^\top \mathbf{1}$ in the money market, where $\mathbf{1} \equiv (1, \ldots, 1)^\top$. The manager's optimization problem, as before, is given by (4).

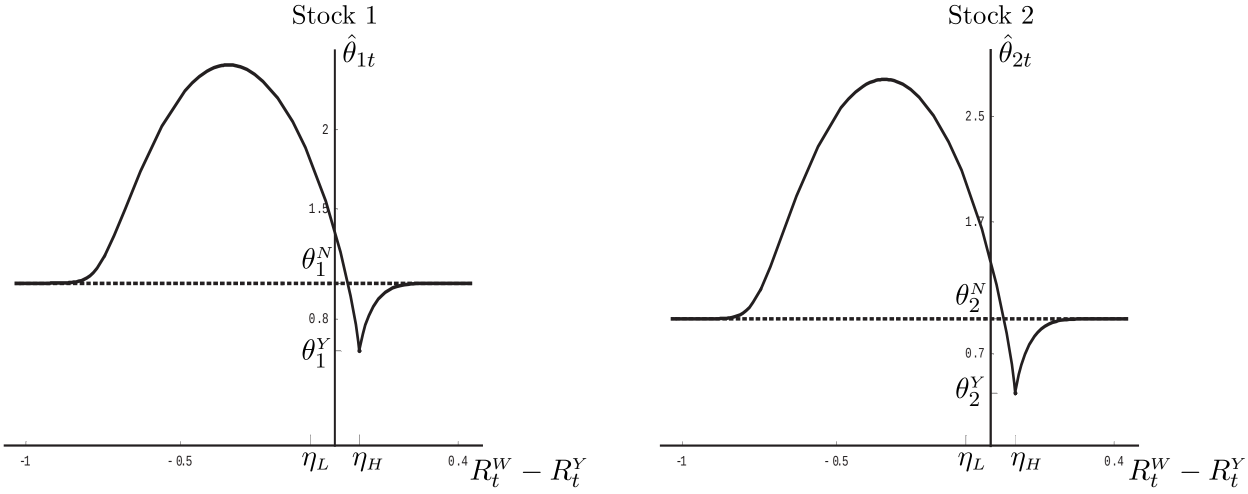
Allowing for multiple sources of uncertainty increases the dimensionality of the problem, introducing certain technical difficulties, and our proof of Proposition 1 does not readily extend. The proof of Proposition 2 (in the Appendix) employs a method to reduce the multi-state variable problem to a single-state variable one via a change of variable (accounting for benchmarking) and a change of measure (accounting for risk aversion). As in Proposition 1, however, we present Proposition 2 in terms of the state-price density process ξ, following $d\xi_t = -\xi_t r dt - \xi_t \kappa^\top dw_t$, where now the market price of risk $\kappa \equiv \sigma^{-1}(\mu - r\mathbf{1})$ is n-dimensional. For brevity, we present only the case where Condition 1 is satisfied.

Proposition 2. *Under Condition 1, the optimal fractions of the manager's portfolio invested in* *risky assets and her terminal wealth are given by*

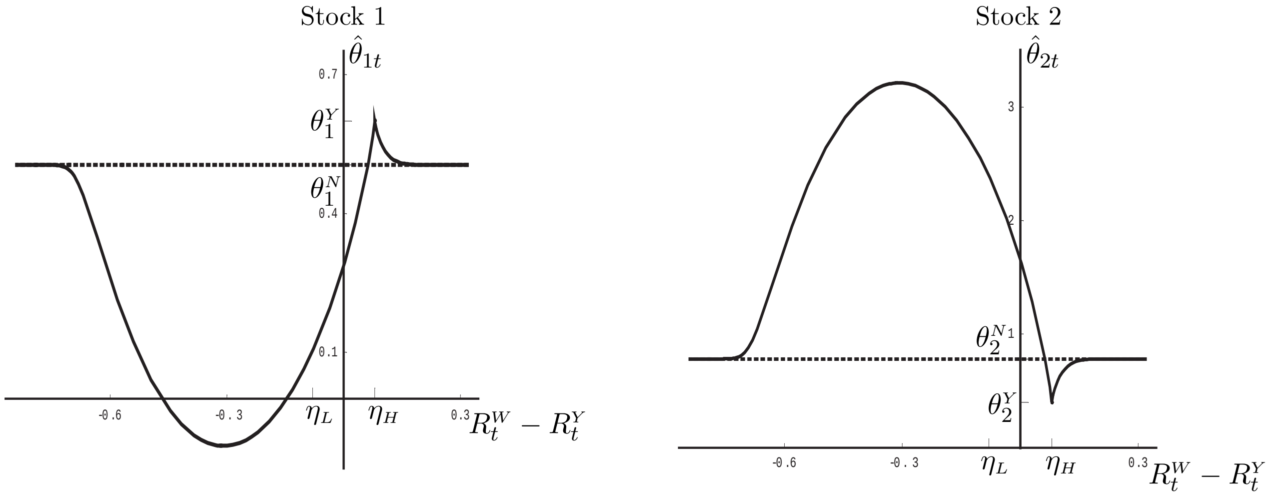
$$\begin{aligned}
\hat{\theta}_t &= \theta^N + (\theta^N - \theta^Y)\Bigg\{\Big[\mathcal{N}(d_2(\gamma,\pi_*)) - \mathcal{N}(d_2(\gamma,\pi^*))\Big]e^{\eta_H} \\
&\quad + \Big[(\pi^{*1/\gamma}\phi(d_1(\gamma,\pi^*)) - \pi_*^{1/\gamma}\phi(d_1(\gamma,\pi_*)))e^{\eta_H} \\
&\quad + (f_H^{(1/\gamma-1)}\phi(d_1(\gamma,\pi_*)) - f_L^{(1/\gamma-1)}\phi(d_1(\gamma,\pi^*)))y^{-1/\gamma}\Big]\frac{\gamma\xi_t^{-1/\gamma}\hat{Z}(\gamma)}{||\kappa^\top - \gamma\sigma^\top\beta||\sqrt{T-t}\,\hat{W}_t}\Bigg\}, \quad (5) \\
\hat{W}_T &= \frac{1}{f_H}J\left(\frac{y}{f_H}\xi_T\right)1_{\{\xi_T Y_T^\gamma < \pi_*\}} + e^{\eta_H}Y_T 1_{\{\pi_* \le \xi_T Y_T^\gamma < \pi^*\}} + \frac{1}{f_L}J\left(\frac{y}{f_L}\xi_T\right)1_{\{\pi^* \le \xi_T Y_T^\gamma\}},
\end{aligned}$$

where y solves $E[\xi_T\hat{W}_T] = W_0$, $J(\cdot)$, $\mathcal{N}(\cdot)$, $\phi(\cdot)$ are as given in Proposition 1, $\pi_* = f_H^{1-\gamma}e^{-\gamma\eta_H}/y$, $\pi^* > \pi_*$ satisfies $\hat{g}(\pi) = 0$, $\hat{g}(\pi) = \left(\gamma\left(\frac{y}{f_L}\pi\right)^{1-1/\gamma} - (f_H e^{\eta_H})^{1-\gamma}\right)/(1-\gamma) + e^{\eta_H}y\pi$, $\hat{Z}(\gamma) = e^{\frac{1-\gamma}{2\gamma^2}||\kappa-\gamma\sigma^\top\beta||^2(T-t)}$, $\hat{d}_1(\gamma,\hat{\pi}) = \frac{\ln\frac{\hat{\pi}}{\xi_t Y_t^\gamma} + \frac{2-\gamma}{2\gamma}||\kappa-\gamma\sigma^\top\beta||^2(T-t)}{||\kappa-\gamma\sigma^\top\beta||\sqrt{T-t}}$, $\hat{d}_2(\gamma,\hat{\pi}) = \hat{d}_1(\gamma,\hat{\pi}) - \frac{1}{\gamma}||\kappa-\gamma\sigma^\top\beta||\sqrt{T-t}$, $\theta^N = \frac{1}{\gamma}(\sigma^\top)^{-1}\kappa$, $\theta^Y = \beta$, and $\hat{W}_t$ is as given in the Appendix. The case of $\theta^N = \theta^Y$ is described in the Appendix.

Proposition 2 reveals that our earlier insights go through component-by-component, which can be viewed as "tilting" positions in individual stocks in response to incentives. Thus, at the flow threshold η_H the manager optimally chooses to select portfolio weights in individual stocks close to those of the benchmark portfolio. When outperforming the benchmark, the manager tilts each portfolio weight away from the benchmark and in the direction of her normal policy, converging to it in the limit. When underperforming, she deviates from the benchmark by tilting the investment in each stock i in the direction dictated by the sign of $\theta_i^N - \theta_i^Y$. For each stock's portfolio weight, we now obtain two typical investment patterns, where the underperforming manager either increases or decreases her weight in the stock, analogous to economies (a) and (b) of Proposition 1. Figure 5 illustrates this for the case of two risky stocks with parameters chosen such that $\theta_1^N > \theta_1^Y$, $\theta_2^N > \theta_2^Y$ (panel (a)) and $\theta_1^N < \theta_1^Y$, $\theta_2^N > \theta_2^Y$ (panel (b)). The remaining cases are mirror images of the ones presented. We note that even with constant security price parameters (r, μ, σ), the manager's optimal investment policy in the presence of incentives does not display two-fund separation. We may rewrite the optimal risk exposure expression in Proposition 2 as $\hat{\theta}_t = (\sigma^T)^{-1}\kappa(1+x_t)/\gamma - \beta x_t$, where x_t denotes the expression in $\{\cdot\}$ of equation (5). Hence, we see that the optimal portfolio satisfies a three-fund separation property, with the three funds being the instantaneous mean-variance efficient portfolio of risky assets $(\sigma^T)^{-1}\kappa/(\mathbf{1}^T(\sigma^T)^{-1}\kappa)$, the benchmark portfolio θ^Y, and the riskless asset.



(a) Economies with $\theta_1^N > \theta_1^Y$ and $\theta_2^N > \theta_2^Y$.



(b) Economies with $\theta_1^N < \theta_1^Y$ and $\theta_2^N > \theta_2^Y$.

Figure 5. The manager's optimal portfolio weights in Stocks 1 and 2. The solid plots are for the optimal portfolio weights, and the dotted plots are for the manager's normal weights. We parameter values are $\gamma = 1.5$, $f_L = 0.8$, $f_H = 1.5$, $\beta = (0.6,\, 0.4)^\top$, $\eta_L = -0.08$, $\eta_H = 0.08$, $\mu = (0.07,\, 0.09)$, $r = 0.02$, $\sigma_1 = (0.18, 0.0)$ (panel (a)), $\sigma_1 = (0.18, 0.08)$ (panel (b)), $\sigma_2 = (0.0, 0.22)$, $t = 0.75$, $T = 1$.

It is also of interest to investigate whether the manager achieves her optimal risk-taking profile by taking on systematic versus idiosyncratic risk, given the current discussion in the literature (e.g., Chevalier and Ellison (1997)). When underperforming, does the manager really take on systematic risk as suggested by our baseline model, or rather, opts for idiosyncratic gambles, more in line with the perceived wisdom? To shed some light on this issue, we present a special case of our model, in which the manager has a choice between systematic and idiosyncratic risks. We consider a two-stock economy with the following parameterization:

$$\mu = \begin{pmatrix} \mu_1 \\ r \end{pmatrix}, \quad \sigma = \begin{pmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{22} \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Here, the relative performance benchmark Y is given by stock 1, which is driven solely by the Brownian motion w_1. In contrast, stock 2 is purely driven by w_2 and does not command any risk premium, with mean return equal to the riskless rate. Consequently, the market price of risk is given by $\kappa = \left(\frac{\mu_1 - r}{\sigma_{11}}, 0\right)^\top$. In other words, by investing in stock 1 the manager is exposed to systematic risk, and by investing in stock 2 the manager takes on idiosyncratic risk. We report the ensuing optimal portfolio weights in stocks 1 and 2 in Figure 6.

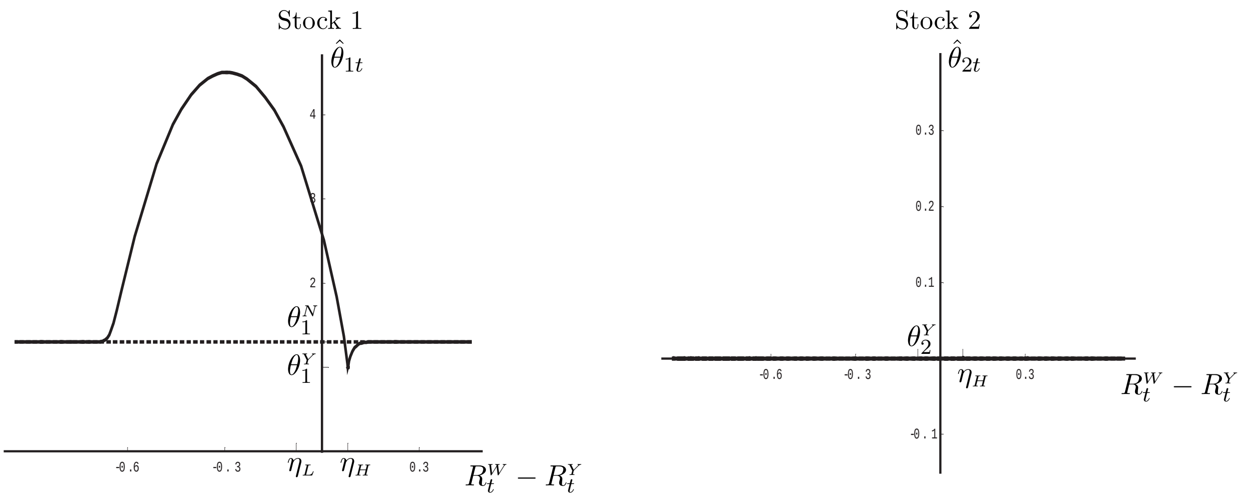


Figure 6. The manager's optimal portfolio weights in Stocks 1 and 2. Stock 1 is driven by systematic risk, while stock 2 is driven by idiosyncratic risk. The solid plots are for the optimal portfolio weights, and the dotted plots are for the manager's normal weights. The parameter values are We parameter values are $\gamma = 1.5$, $f_L = 0.8$, $f_H = 1.5$, $\beta = (1.0,\, 0.0)^\top$, $\eta_L = -0.08$, $\eta_H = 0.08$, $\mu = (0.07,\, 0.02)$, $r = 0.02$, $\sigma_1 = (0.16, 0.0)$ $\sigma_2 = (0.0, 0.22)$, $t = 0.75$, $T = 1$.

As evident from Figure 6, the manager chooses to optimally invest *nothing* in stock 2, and hence is not exposed to any idiosyncratic risk. Instead, she is only exposed to systematic risk and engages in optimal risk-shifting via adjusting her position in stock 1, in a manner consistent with our results of Section 2. We note that the above example is not pathological. We have considered a sequence of economies parameterized by μ_2^m, with $\mu_2^m \to r$ as $m \to \infty$. In each economy m, the market price of risk due to w_2 is nonzero, and the weights in each risky asset are similar to those presented in Figure 6. As we increase m, the investment in the second stock uniformly converges to zero.

4. Empirical Analysis

Some support for our model's implications can be drawn from the extant empirical literature on risk-shifting behavior of mutual funds. However, existing empirical evidence is insufficient to verify the central implications of our model, which are novel. Namely, no empirical paper drew a distinction between economies of types (a) versus (b). In the latter, according to the model, the manager's gambling entails reducing the volatility (systematic risk) of her portfolio when underperforming. Moreover, the measure of risk-taking incentives arising from our model is the deviation of the manager's portfolio from the benchmark − the manager's portfolio *tracking error variance* $\text{Var}(R_t^W - R_t^Y)$ − and not the portfolio variance $\text{Var}(R_t^W)$. An underperforming manager is predicted to always increase the tracking error variance, while the variance may or may not increase in response to incentives.

The existing literature has been sparked by Brown, Harlow, and Starks (1996) who find that underperforming managers increase the volatility of their portfolios towards the year-end. However, there remains some controversy about this result, stemming from Busse (2001), who finds no such increase. Nevertheless, the papers above offer tests of our model only if the world is of type (a), since the measure of portfolio riskiness adopted in these studies is the portfolio's total variance, and not the tracking error variance. Chevalier and Ellison (1997) also find that managers with low January through September excess returns have bigger risk-taking incentives towards the year-end than of those with high excess returns. Their paper, in contrast, can be viewed as providing stronger support for our conclusions as they do use the sample variance of a fund's excess return over the market as a measure of the fund's riskiness. However, they estimate January through September and September through December standard deviations on monthly data, and hence the estimates could be quite imprecise. Also related is recent empirical work by Reed and Wu (2005). Reed and Wu examine risk-shifting incentives of mutual fund managers in response to beating the benchmark and beating their peers using daily data on mutual fund returns and holdings. Employing a broad set of tests, they find strong evidence that performance relative to the S&P 500 index is a significant determinant of funds' risk-taking choices while performance relative to each other is not. Mutual fund managers appear to take larger risks before beating the S&P 500; after beating they tend to increase the correlation of their funds' assets with the market. None of these papers looks at risk-taking behavior of managers who normally desire lower risk exposure than their benchmark (our economies (b)).

We here provide a set of simple empirical tests that focus on the implications novel to our analysis, and thus complement the existing body of work. We combine daily returns data on US mutual funds from the International Center for Finance at Yale SOM[20] with CRSP mutual-fund database. Using CRSP objective codes, we leave only actively managed US equity mutual funds

[20]We are indebted to Will Goetzmann and Geert Rouwenhorst for making these data available to us.

in the aggressive growth, growth and income, and long-term growth categories. The data ranges from 1970 to 1998. This data range is comparable to those used by Chevalier and Ellison and Sirri and Tufano (1998), who document convexities in the flow-performance relationship, required for our risk-taking implications.

Our first hypothesis is that the tracking error variance is higher for managers whose year-to-date returns are below that of the benchmark than for those whose return are above the benchmark. To examine this hypothesis, for each year, we use funds' daily returns to estimate monthly sample standard deviations of each fund's *excess* return—a total of 12 entries per fund per year. For comparison, we also estimate monthly sample standard deviations of fund returns, as in Busse. On a monthly basis, for each fund, we compute an OVER indicator that equals one if the fund was overperforming the S&P 500 index, a benchmark also used by Chevalier and Ellison and others, as of the end of the preceding month. To reduce the noise in the data coming from the funds that are right at the benchmark-beating boundary, we require that a fund's year-to-date return was above the market for four out of last five trading days of the preceding month. Finally, we drop fund-year entries with fewer than 200 daily observations and fund-year entries for funds that were either above or below the S &P 500 for 90% of the year, displaying no transition we are interested in. We then regress the tracking error standard deviation, our risk measure, on the OVER indicator. The regression includes fund-year fixed effects.[21] Inclusion of such fixed effects is very conservative and perhaps does not do justice to the richness of our data set, where we could have explored the effects of interesting control variables, varying over time and across funds. But to keep this section brief, we maintain our focus on the predictions our theory and look at the the within-year variation of managers' policies. For comparison, we repeat this exercise for standard deviation of fund returns as the dependent variable.

Table 2 reveals that the OVER indicator comes out significant in the tracking error regression. Consistent with our theory, managers tend to reduce their tracking error variance when overperforming the market, and gamble by increasing it when underperforming. In contrast, the effect on the variance of fund returns has the opposite sign, however, the effect becomes insignificant throughout many of our robustness checks. This finding is in line with Busse, who argues that underperforming managers do not seem to manipulate their portfolio standard deviation towards the year end. We have performed a number of robustness checks and the message was the same.[22]

[21] We thank Roberto Rigobon and Antoinette Schoar for suggesting this approach to us.

[22] We have repeated the exercise with an alternative, more contemporaneous, measure for the OVER indicator, OVER1, which equals one if the fund is overperforming the S&P 500 index for more than three quarters of the concurrent month, and zero otherwise. We have also tried the lagged version of it as a possible alternative. Throughout all these tests the effect of overperforming the S&P 500 index was always negative and significant at the 1% level. We have also repeated the exercise winsorizing our fund returns data to reduce the impact of outliers and data errors. Additionally, we have put lagged dependent variable in the regression, as were concerned about autocorrelation in the dependent variable, and the results were very similar to those in Table 2. Finally, we have repeated the regressions clustering by month and fund objective code, guarding against potential problems due to cross-sectional correlation. Significance of all of our possible OVER indicators has dropped, but still remained at the 5% level. In the standard deviation regressions, however, all possible OVER indicators became insignificant.

	Dependent Variable:	
	$\sigma_m(R^W_{i,t} - R^Y_t)$	$\sigma_m(R^W_{i,t})$
$\text{OVER}_{i,m} \times 10^3$	-0.1819 (-4.73)	0.1058 (2.38)
Fund-year fixed effects	Yes	Yes
R^2	0.39	0.36
Number of observations	40721	40721

Table 2. Tracking error standard deviation and fund return standard deviation tests. $R^W_{i,t}$ is fund i's return on day t of month m, R^Y_t is the return of the S&P 500 index on day t of month m, $\text{OVER}_{i,m}$ is an overperformance indicator for month m defined above, and $\sigma_m(\cdot)$ is the sample standard deviation for month m. t-statistics are in parentheses. All standard errors are corrected for heteroscedasticity.

Our second set of tests aims at providing evidence for economies of type (b), in which a manager gambles through reducing the systematic risk of her portfolio, rather than increasing it. Recall that, according to our model, a manager would choose this risk exposure pattern if she is sufficiently risk averse, i.e., normally chooses a portfolio that is less risky than her benchmark. To identify such managers, we look at (i) their portfolio betas in the preceding year, (ii) their betas in the current year, and construct a subsample of funds whose betas are below one. Then, for each fund for each week, we construct for an UNDER indicator that equals one when the fund's year-to-date return is below that of the S&P 500 as of the end of the preceding week. To reduce noise coming from the funds on the benchmark-beating boundary, we require that additionally the fund was above the S&P 500 for four out of five days of the preceding week. Our hypothesis is that sufficiently risk-averse managers decrease their portfolio betas when underperforming the market. Of course, there are many other influences potentially affecting funds' betas. We therefore also include fund-year fixed effects, as well as month fixed effects. The latter are to control for seasonality in betas, documented by, e.g., Lewellen and Nagel (2005). Because of the computational restriction to keep the size of the dummy-variable set manageable, we do not include any fixed effects for the intercept terms, but those are very small and insignificant to make any appreciable difference. We again drop fund-years with fewer than 200 entries and drop fund-years in which the fund displayed no transition in its year-to-date relative return for over 90% of the year. We then regress funds' excess returns on the excess return on the S&P 500 interacted with the UNDER indicator as well as the month and fund dummies, and report the results in Table 3.

	Dependent Variable: $R^W_{i,t} - R^F_t$			
	Beta$_T$ below 1		Beta$_{T-1}$ below 1	
	(1)	(2)	(3)	(4)
UNDER$_{i,w} \times (R^Y_t - R^F_t)$	-0.017	-0.020	-0.018	-0.021
	(-6.61)	(-7.72)	(-6.99)	(-8.22)
Month fixed effects	No	Yes	No	Yes
Fund-year fixed effects	Yes	Yes	Yes	Yes
R^2	0.37	0.37	0.37	0.37
Number of observations	808642		816783	

Table 3. Beta tests. $R^W_{i,t}$ is fund i's return on day t R^Y_t is the return of the S&P 500 index on day t, R^F_t is the riskfree interest rate on day t, and UNDER$_i$ is an underperformance indicator for week w defined above. The fixed effects dummies are interacted with $(R^Y_{i,t} - R^F_t)$. t-statistics are in parentheses.

In both tests, the underperformance indicator interacted with the market is negative and significant. Underperforming managers who fall into our type-(b) category, i.e., whose betas are below one, choose to reduce their portfolio betas. The effect is similar in magnitude and significance for the two approaches to isolating managers who normally desire lower risk exposure than the market: based on their betas this year or last year. We have performed a number of robustness checks and the message was the same.[23] However, while the signs of our estimates are consistent with our theory, their magnitudes are considerably smaller, most likely reflecting the risk-management constraints imposed in practice (see Almazan, Brown, Carlson, and Chapman (2004)).

5. Conclusion

In this paper we have attempted to isolate the two most important adverse incentives of a fund manager: an implicit incentive to perform well relative to a benchmark, and an explicit incentive to manage the fund in accordance with her own appetite for risk. These incentives introduce a nonconcavity in the manager's problem, akin to nonconcavities observed in many corporate finance

[23] We have tried alternative definitions of the UNDER indicator, basing it solely on performance as of Friday of the preceding week, or just on the preceding week, in which case UNDER equals one when the fund was trailing the S&P 500 index for four days or more of the preceding week. Additionally, we have put several lags of the dependent variable in the regression, as were concerned about autocorrelation in funds daily returns. The lagged dependent variable was highly significant, but the significance of the interaction term with UNDER was virtually unchanged. Due to the large size of our sample, we have been able to implement White correction for heteroscedasticity only on random 35% subsamples of funds in our dataset. Additionally, in these subsamples we have clustered errors by day and fund objective code. In all 10 random subsamples of funds, our result remained significant at the 5% level.

applications (e.g., asset substitution problem, "gambling for resurrection," executive compensation, hedge fund managers compensation). It has been argued that in some of these applications, agents do not behave as though they are risk-neutral, and may be effectively risk averse. Our methodology of dealing with nonconcavities in the presence of risk aversion may then help shed some light on these and other issues of interest. We solve the manager's problem within a standard dynamically-complete Black and Scholes (1973) framework. The complete markets assumption offers considerable tractability, allowing us to derive the manager's optimal policy in closed-form, and also establishes a useful benchmark reference point for future research. In many real world applications, nonconcavities in the payoff structure go hand-in-hand with capital markets imperfections, frictions, for example with restrictions against trading the underlying security designed to induce the "right" incentives to the manager. Our story of the optimal interaction of risk-shifting with risk aversion would then be further compounded by the effects of such frictions.

In our setup, the need to constrain the manager's investment choice arises naturally as the potential costs of misaligned incentives resulting from the manager's policy are shown to be economically significant. We believe that investigating investment restrictions that may ameliorate the adverse effects of managerial incentives is a fruitful area for future research. It would also be of interest to endogenize within our model the fund-flows to relative-performance relationship that we have taken as given. Finally, while we focus on the fund manager's changing her portfolio composition in response to incentives, this may not be the only way the manager can increase her relative performance. As Christoffersen (2001) documents, about half of money fund managers voluntarily waive fees so as to increase their net return, and variation in these fee waivers is largely driven by relative performance. It would be interesting to incorporate such a feature into our model.

Appendix A: Proofs

Proof of Proposition 1. Before proceeding with the proof, we present for completeness the results that for brevity were not included in the body of the proposition. First, note that for $\gamma = 1$, $g(\cdot)$ takes the form: $g(\xi) = -\left(\ln\frac{\hat{y}\xi}{f_L} + 1 - \ln\frac{\xi^{1/\hat{\kappa}}}{Af_H}\right) + yA\xi^{1-1/\hat{\kappa}}$. Second, since $\hat{W}_t\xi_t$ is a martingale (given the dynamics of $\hat{W}_t$ and ξ_t), the time-t wealth is obtained by evaluating the conditional expectation of $\hat{W}_T\xi_T$. In the economies described in (a):

$$\begin{aligned}\hat{W}_t &= E_t\left[\hat{W}_T\xi_T/\xi_t\right] \\ &= \left[\mathcal{N}(d(\gamma,\hat{\xi}))f_H^{(1/\gamma-1)} + \mathcal{N}(-d(\gamma,\xi_a))f_L^{(1/\gamma-1)}\right] Z(\gamma)(\hat{y}\xi_t)^{-1/\gamma} \\ &\quad + \left[\mathcal{N}(d(\hat{\kappa},\xi_a) - \mathcal{N}(d(\hat{\kappa},\hat{\xi}))\right] A\, Z(\hat{\kappa})\xi_t^{-1/\hat{\kappa}}\,. \end{aligned} \tag{A1}$$

Similarly, in the economies described in (b):

$$\begin{aligned}\hat{W}_t &= \left[\mathcal{N}(d(\gamma,\xi_b))f_L^{(1/\gamma-1)} + \mathcal{N}(-d(\gamma,\hat{\xi}))f_H^{(1/\gamma-1)}\right] Z(\gamma)(\hat{y}\xi_t)^{-1/\gamma} \\ &\quad + \left[\mathcal{N}(d(\hat{\kappa},\hat{\xi}) - \mathcal{N}(d(\hat{\kappa},\xi_b))\right] A\, Z(\hat{\kappa})\xi_t^{-1/\hat{\kappa}}\,. \end{aligned} \tag{A2}$$

Finally, when $\theta^N = \theta^Y$, if $\eta_H \leq 0$ (so that $e^{\eta_H}Y_T$ and hence $R_T^Y + \eta_H$ are feasible) or $\eta_H \geq \tilde{\eta}$ (so that $e^{\eta_H}Y_T$ and hence $R_T^Y + \eta_H$ are above a critical level of infeasibility), then $\hat{\theta}_t = \theta^N$ and $\hat{W}_T = J(\hat{y}\xi_T)$; otherwise $\hat{W}_T = \left\{\frac{1}{f_L}J\left(\frac{\bar{y}}{f_L}\xi_T\right) \;\; or \;\; e^{\eta_H}Y_T\right\}$, with the indifference solution alternating between the two values in any way that satisfies the budget constraint, where (using the steps outlined below) one can show that $\tilde{\eta}$ solves $g\left(\xi = 1, y = f_L^{1-\hat{\kappa}}e^{(1-\hat{\kappa})(r+\beta(\mu-r)/2)T}/W_0^{\hat{\kappa}}, \eta_H\right) = 0$, and $\bar{y} > f_H^{1-\gamma}/A^\gamma$ solves $g(\xi = 1, y) = 0$.

We now proceed with the steps of the proof. To obtain the risk exposure expressions in the proposition, note that from (1), the diffusion term of the manager's optimal portfolio value process is $\hat{\theta}_t\sigma\hat{W}_t$. Equating the latter term with the diffusion term obtained by applying Itô's Lemma to (A1) and (A2) yields the expressions for $\hat{\theta}_t$ in economies (a) and (b), respectively. Therefore, to complete the proof, it is sufficient to establish optimality of the given terminal wealth $\hat{W}_T$. Methodologically, most related to this proof is the proof of Proposition 2 of the constrained model in Basak, Shapiro, and Tepla (2004), but the setting here is notably different and the optimization problem is unconstrained. The non-concavity of the problem, arising due to (4), has an extra dimension of complexity, as not only does the expression we employ for the convex conjugate has a discontinuous non-concavity at a stochastic location, but the magnitude of the discontinuity is stochastic as well. Therefore, to our knowledge, the way the proof below adapts the martingale representation and the convex-duality techniques (see, e.g., Karatzas and Shreve (1988)) to a non-concave problem, has not been previously used in the literature.

Appealing to the martingale representation approach, the dynamic budget constraint (1) of the manager's optimization problem can be restated using the terminal value of the state price density process as $E[\hat{W}_T \xi_T] = W_0$. The manager thus effectively solves a static variational problem (in which, without loss of generality, $\xi_0 = 1$). For a given value of θ^N and θ^Y, let W_T be any candidate optimal solution, satisfying the static budget constraint $E[W_T \xi_T] \leq W_0$. Consider the following difference in the manager's expected utility:

$$\begin{aligned} & E[u(\hat{W}_T f_T)] - E[u(W_T f_T)] = E[u(\hat{W}_T f_T)] - \hat{y} W_0 - (E[u(W_T f_T)] - \hat{y} W_0) \\ & \geq E[u(\hat{W}_T f_T)] - E[\hat{y} \hat{W}_T \xi_T] + (E[u(W_T f_T)] - E[\hat{y} W_T \xi_T]) = E[v(\hat{W}_T, \xi_T) - v(W_T, \xi_T)] \,, \end{aligned} \tag{A3}$$

where the inequality is due to $\hat{W}_T$ satisfying the budget constraint with equality, while W_T satisfying the budget constraint with inequality, and where

$$v(W, \xi) = u(W f_L 1_{\{R^W - R^Y < \eta_H\}} + W f_H 1_{\{R^W - R^Y \geq \eta_H\}}) - \hat{y} W \xi \,. \tag{A4}$$

To show optimality of $\hat{W}_T$, it is left to show that the right-hand side of (A3) is non-negative. Under the geometric Brownian motion dynamics of Y_T and ξ_T, and using the normalization of Y_0 and ξ_0, it is straightforward to verify that $Y_T = A e^{-\eta_H} \xi_T^{-\beta\sigma/\kappa}$. The expression in (A4) is thus simplified to

$$v(W, \xi) = u(W f_L 1_{\{W < A\xi^{-\beta\sigma/\kappa}\}} + W f_H 1_{\{W \geq A\xi^{-\beta\sigma/\kappa}\}}) - \hat{y} W \xi \,. \tag{A5}$$

Given the manager's CRRA preferences, to establish the non-negativity of (A3), one needs to account for the relation between the parameters γ, β, σ, and κ in (A5). To avoid repetition of technical details, we provide the proof for optimality of $\hat{W}_T$ for the economies in (a) with $\gamma > 1$. The logic of the proof applies to the remaining subdivisions of the parameter space, as identified in the Proposition. Therefore, we now show that for the case in which $\kappa/(\beta\sigma) > \gamma > 1$,

$$\arg\max_W v(W, \xi) = f_H^{1/\gamma - 1} (\hat{y}\xi)^{-1/\gamma} 1_{\{\xi < \hat{\xi}\}} + A \xi^{-\beta\sigma/\kappa} 1_{\{\hat{\xi} \leq \xi < \xi_a\}} + f_L^{1/\gamma - 1} (\hat{y}\xi)^{-1/\gamma} 1_{\{\xi_a \leq \xi\}} \,.$$

Indeed, in the above convex conjugate construction, there are three local maximizers of $v(W, \xi)$: $W_H \equiv \frac{1}{f_H} J\left(\frac{\hat{y}}{f_H} \xi\right) = f_H^{1/\gamma - 1} (\hat{y}\xi)^{-1/\gamma}$, $W_L \equiv \frac{1}{f_L} J\left(\frac{\hat{y}}{f_L} \xi\right) = f_L^{1/\gamma - 1} (\hat{y}\xi)^{-1/\gamma}$, and $W_A \equiv A\xi^{-\beta\sigma/\kappa}$, where each of the three can become the global maximizer of $v(W, \xi)$ for different values of ξ. When $\xi = \hat{\xi}$, then $W_H(\hat{\xi}) = W_A(\hat{\xi})$. When $\xi < \hat{\xi}$, then $W_L > W_H > W_A$ holds under the given subdivision of the parameter space, and so for $W \in \{W_L, W_H, W_A\}$, we get $v(W, \xi) = u(W f_H) - \hat{y} W \xi$, establishing W_H as the global maximizer. When $\xi > \hat{\xi}$, then $W_H < \min(W_A, W_L)$, and $v(W_H, \xi) = u(W_H f_L) - \hat{y} W_H \xi$, establishing that W_H cannot be the global maximizer, because for $W_L < W_A$ by the local optimality of W_L, we have $u(W_H f_L) - \hat{y} W_H \xi < u(W_L f_L) - \hat{y} W_L \xi = v(W_L, \xi)$, and for $W_L \geq W_A$ accounting for the local optimality of W_A due to the stochastic non-concavity, we have $u(W_H f_L) - \hat{y} W_H \xi < u(W_A f_L) - \hat{y} W_A \xi = v(W_A, \xi)$. Moreover, for $\xi > \underline{\xi}$, where $\underline{\xi} = (\hat{y} A^\gamma / f_L^{1-\gamma})^{1/(\gamma/\hat{\kappa} - 1)} > \hat{\xi}$, we have $W_L < W_A$; whereas for the range $\hat{\xi} \leq \xi < \underline{\xi}$, we get $W_L > W_A$, and for this range W_A is the global maximizer. Finally, note that for $\xi > \underline{\xi}$, we obtain $v(W_L, \xi) = v(W_A, \xi) + g(\xi)$. Then, using $\kappa/(\beta\sigma) > \gamma > 1$, and the fact that $g(\underline{\xi}) < 0$, $g(\infty) = \infty$, it is straightforward to verify that $g(\xi) > 0$ if and only if $\xi > \xi_a$, where $g(\xi_a) = 0$, and $\xi_a > \underline{\xi}$, thereby completing the proof for the case of interest in the parameter space. Note that since $W_L(\underline{\xi}) = W_A(\underline{\xi})$, having $\hat{W} = W_A$ for $\hat{\xi} < \xi < \xi_a$ gives rise to a discontinuity in $\hat{W}_T$ as a function of ξ_T at ξ_a. The discontinuity arises in the other subcases in (a) as well, and analogously, under the parameter values in (b), the optimal policy is discontinuous at ξ_b. ∎

Proof of Proposition 2. To establish optimality of the stated terminal wealth for the manager's optimization problem in (4) under n sources of uncertainty and n risky assets, it is still sufficient to establish the non-negativity of the right-hand side of (A3), as in the proof of Proposition 1. However, because for $n > 1$, ξ and Y each span a different subspace of the state space, $v(W, \xi)$ as given in (A4) does not in general simplify to the parametric form in (A5). Therefore, to gain further tractability, note that in (A4), $v(W, \xi) = Y^{1-\gamma}\hat{v}(\frac{W}{Y}, \xi Y^{\gamma})$, where $Y^{1-\gamma} \geq 0$, and

$$\hat{v}(V, \pi) = u(V f_L 1_{\{V < e^{\eta_H}\}} + V f_H 1_{\{V \geq e^{\eta_H}\}}) - y\pi V\,.$$

Using this change of variables ($V \equiv \frac{W}{Y}$, $\pi \equiv \xi Y^{\gamma}$), one obtains:

$$\begin{aligned} \arg\max_V \hat{v}(V, \pi) &= f_H^{1/\gamma - 1}(y\pi)^{-1/\gamma} 1_{\{\pi < \pi_*\}} + e^{\eta_H} 1_{\{\pi_* \leq \pi < \pi^*\}} + f_L^{1/\gamma - 1}(y\pi)^{-1/\gamma} 1_{\{\pi^* \leq \pi\}} & \text{(A6)} \\ &= \left[\frac{1}{f_H} J\left(\frac{y}{f_H}\xi\right) 1_{\{\pi < \pi_*\}} + e^{\eta_H} Y_T 1_{\{\pi_* \leq \pi < \pi^*\}} + \frac{1}{f_L} J\left(\frac{y}{f_L}\xi\right) 1_{\{\pi^* \leq \pi\}}\right] \frac{1}{Y}\,. & \text{(A7)} \end{aligned}$$

The equality in (A6) is readily verified by following steps analogous to those in Proposition 1, to show that each of the three local maximizers in the above convex conjugate construction is indeed a global maximizer in designated ranges of π. The equality in (A7) holds due to the change of variables. The expression in brackets in (A7) is the terminal wealth, $\hat{W}$, stated in the Proposition, thereby verifying its optimality. Note from (A6)–(A7) that $\hat{V}_T \equiv \frac{\hat{W}_T}{Y_T}$ is a function of only $\pi_T \equiv \xi_T Y_T^{\gamma}$. Also note that the diffusion component of $\xi_T Y_T^{\gamma}$ is a function of

$$-\kappa^\top w_T + \gamma\beta^\top \sigma w_T = -\gamma w_T^\top \sigma^\top (\frac{1}{\gamma}(\sigma^\top)^{-1}\kappa - \beta) = -\gamma w_T^\top \sigma^\top (\theta^N - \theta^Y).$$

The latter indicates that the optimal policy is driven by the component-wise relation between the manager's normal weights in each stock, θ_i^N, compared with the benchmark weights θ_i^Y, $i = 1, \ldots, n$. With $n = 1$, Proposition 1, for expositional purposes, separately examines economies (a) ($\theta^N < \theta^Y$) and (b) ($\theta^N > \theta^Y$), to highlight the economic intuition of each case. With $n > 1$, Proposition 3 does not refine the expressions to account for all possible relations between γ, β, σ, and κ. We present results in their general form, and discuss how the basic intuition, with $n = 1$, extends to the case where various components of the optimal policy behave according to their counterparts in economies (a) or (b). Similarly, when for some i, $\theta_i^N = \theta_i^Y$, then $\hat{\theta}_i = \theta_i^N$. However, in the case where $\theta_i^N = \theta_i^Y$, for *all* i, then as discussed in Proposition 1, $\hat{W}_T$ behaves either as in the normal case, or alternates between the normal and the benchmark levels.

Given the optimal terminal wealth, we proceed to derive the wealth dynamics and the trading strategy. Relying on the martingale property of ξW, $\hat{W}_t = E_t\left[\hat{W}_T \xi_T / \xi_t\right]$, but unlike in Proposition 1, one should account here for the joint distribution of Y^{γ} and ξ, under the physical probability measure P. To circumvent this, it is helpful to employ a change of measure. The new measure, G, is defined by a Radon-Nikodym derivative, which accounts for the manager's risk aversion and the composition of the benchmark:

$$\frac{dG}{dP} \equiv e^{-\frac{1}{2}(1-\gamma)^2||\beta^\top\sigma||^2 T+(1-\gamma)\beta^\top\sigma\, w_T}\,.$$

Combining this change of measure with the change of variables, the martingale property of ξW implies that

$$\hat{V}_t \equiv \frac{\hat{W}_t}{Y_t} = E_t\left[\frac{\xi_T Y_T}{\xi_t Y_t}\hat{V}(\pi_T)\right] = E_t^G\left[\rho_t\frac{\pi_T}{\pi_t}\hat{V}(\pi_T)\right]\,, \tag{A8}$$

where E^G is the expectation under the new measure, and $\rho_t = e^{(1-\gamma)(r+\beta^\top\sigma\,\kappa-\gamma||\beta^\top\sigma||^2/2)(T-t)}$ is a deterministic function of time. The first equality in (A8) emphasizes the aforementioned fact that $\frac{\hat{W}_T}{Y_T}$ is only a function of π_T (π_T is log-normally distributed). The second equality in (A8) follows using $Y_T = Y_t e^{(r+\beta^\top\sigma\,\kappa-||\beta^\top\sigma||^2/2)(T-t)+\beta^\top\sigma\,(w_T-w_t)}$ and $w_t = (1-\gamma)\sigma^\top\beta\, t + w_t^G$, where w_t^G is an n-dimensional standard Brownian motion under G. After restating the problem in terms of π, the time-t wealth is straightforwardly obtained by evaluating the conditional expectation in (A8) under G:

$$\begin{aligned}\hat{W}_t &= Y_t\, E_t^G\left[\pi_T\hat{V}(\pi_T)\,(\rho_t/\pi_t)\right]\\ &= \left[\mathcal{N}(\hat{d}_1(\gamma,\pi_*))f_H^{(1/\gamma-1)} + \mathcal{N}(-\hat{d}_1(\gamma,\pi^*))f_L^{(1/\gamma-1)}\right]\hat{Z}(\gamma)(y\xi_t/\rho_t)^{-1/\gamma}\\ &\quad + \left[\mathcal{N}(\hat{d}_2(\gamma,\pi^*)) - \mathcal{N}(\hat{d}_2(\gamma,\pi_*))\right]e^{\eta_H}Y_t\,.\end{aligned} \tag{A9}$$

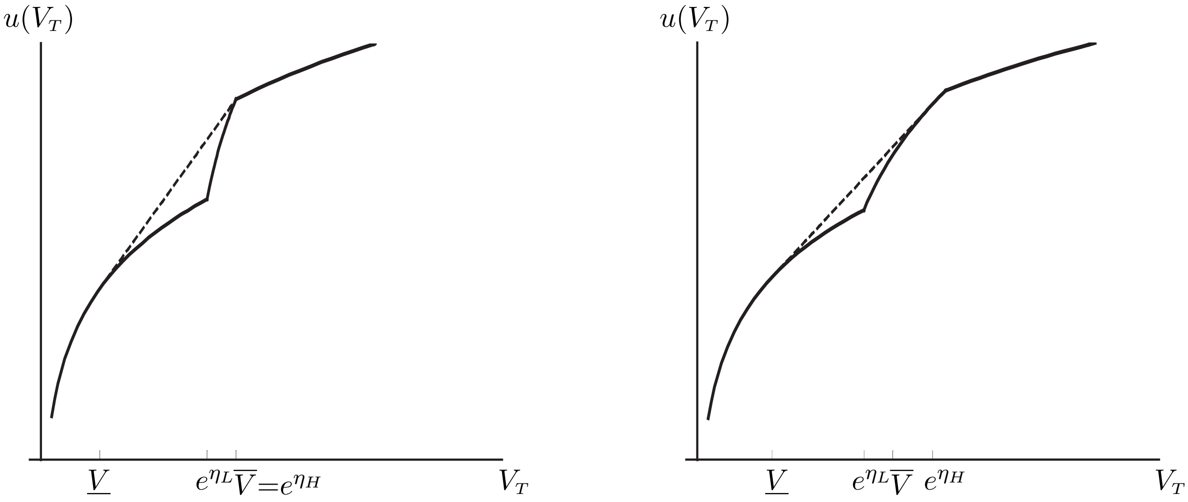
Note from (A9) that $\frac{\hat{W}_t}{Y_t}$, and hence $R_t^W - R_t^Y$, is a function of only $\pi_t \equiv \xi_t Y_t^\gamma$.

The expression for $\hat{\theta}_t$ in the Proposition follows directly from an application of Itô's Lemma to (A9). It is helpful to note the fact that like $\frac{\hat{W}_t}{Y_t}$, $\hat{\theta}_t$ is also a function of only π_t, which is evident since $\hat{\theta}_t$ depends on π_t via the $\hat{d}$ terms and via $\frac{\xi_t^{-1/\gamma}}{\hat{W}_t} = (\xi_t Y_t^\gamma)^{-1/\gamma}\frac{Y_t}{\hat{W}_t} = \pi_t^{-1/\gamma}/\hat{V}(\pi_t)$. Finally, we observe that (A8) can be restated as

$$\hat{V}_t(\pi_t) = \int_{-\infty}^{\infty}\rho_t e^{h_1(t,z)}\hat{V}_T(\pi_t e^{h_1(t,z)})h_2(t,z)dz\,,$$

where, $h_1(t,z) = e^{-\frac{1}{2}||k-\gamma\sigma^\top\beta||(T-t)-z} \geq 0$, $h_2(t,z)\geq 0$ is a probability density function of a normal random variable with zero mean and a variance that is a deterministic function of time, and $\hat{V}_T(\cdot)$ is the non-increasing function in (A6) with a zero derivative over $[\pi_*,\pi^*)$ and a negative derivative over $(0,\pi_*)\cup(\pi^*,\infty)$. Differentiating under the integral with respect to π_t (assuming appropriate regularity conditions), establishes that $\forall\, 0<t<T$, $\hat{V}_t(\pi_t)$ is a monotonically decreasing function of π_t. Therefore, regardless of the dimensionality of the structure of uncertainty in the economy, we can use the isomorphism between π_t and $\frac{\hat{W}_t}{Y_t}$ to plot $\hat{\theta}_t$ as a function of $R_t^W - R_t^Y$. ■

Appendix B: Nonconcavities and Risk Shifting



(a) Condition 1 satisfied.

(b) Condition 1 violated.

Figure 7. The manager's terminal utility. The solid plots are for the manager's utility as given by (3), with its concavification superimposed with the dashed line, where $\underline{V}$, $\overline{V}$ denote the concavification points and $V_T = W_T/Y_T$ denotes the relative portfolio value. Condition 1 is violated when $\overline{V} < e^H$.

To further illustrate the role of nonconcavities, we consider a simple static setting with two periods (0, T). As shown in the proof of Proposition 2, via a change of variable and a change of measure one can replace the manager's objective function by that defined over the relative portfolio value $V_T = W_T/Y_T$, and replace the budget constraint by an equivalent representation in terms of V_t. In particular, for the collar-type fund flow specification (2), the manager's objective function becomes:

$$u(V_T) = \begin{cases} u_L(V_T) & \equiv & (V_T f_L)^{1-\gamma}/(1-\gamma) & \text{if} & V_T < e^{\eta_L}, \\ u_M(V_T) & \equiv & (V_T f_L + \psi V_T(\log V_T - \eta_L))^{1-\gamma}/(1-\gamma) & \text{if} & e^{\eta_L} \geq V_T < e^{\eta_H}, \\ u_H(V_T) & \equiv & \Big(V_T f_H\Big)^{1-\gamma}/(1-\gamma) & \text{if} & V_T \geq e^{\eta_H}. \end{cases}$$

The concavification of this function involves solving for the concavification points $\underline{V}$ and $\overline{V}$ and the chord between $\underline{V}$ and $\overline{V}$. Formally, this requires solving the system of equations

$$u_L(\underline{V}) = a + b\underline{V}, \qquad u_M(\overline{V}) = a + b\overline{V}, \tag{A10}$$

$$u'_L(\underline{V}) = b, \qquad u'_M(\underline{V}) = b, \tag{A11}$$

for a, b, $\underline{V}$, and $\overline{V}$. The tangency point $\overline{V}$ may or may not belong to the range (e^{η_L}, e^{η_H}). If it does, the concavification occurs within the range where $u(\overline{V})$ is given by $u_M(\overline{V})$. Otherwise, the concavification occurs at the corner, at e^{η_H}. The condition distinguishing the latter case is

$\overline{V} \geq e^{\eta_H}$, which is equivalent to Condition 1 (after some algebra). If Condition 1 holds, equations (A10)–(A11) need to be modified accordingly to account for the corner solution.

Figure 7 depicts the transformed utility function $u(V_T)$ with its concavification superimposed on the plot by the dashed line. In both panels, one can see a finite range of relative portfolio values over which the utility is nonconcave, and hence the manager has an incentive to gamble. That is, she would always prefer adding a zero present value gamble {$+\varepsilon$ with probability 50%, -ε with probability 50%} to her status quo portfolio (defined by some fixed $\hat{\theta}$) over ending up with a value in the suboptimal range $(\underline{V}, \overline{V})$. Indeed, the optimal terminal fund portfolio value derived in Proposition 1 features a discontinuity, responsible for the manager's risk-shifting behavior discussed in Section 2.2. Figure 7 also highlights the role of Condition 1. In panel (a), the intermediate segment of the objective function is subsumed within the suboptimal range, and hence its shape is immaterial for the manager's choice of her optimal terminal portfolio value and hence the risk exposure. In panel (b), part of the intermediate segment of the objective function $u_M(\cdot)$ falls outside the suboptimal range, and hence the functional form defining this segment enters in the solution of the manager's optimal policy.

How would the manager, whose investment opportunity set consists of assets with continuous distributions, achieve a discontinuous optimal wealth profile? Simply, by taking advantage of continuous trading and thus synthetically replicating a 50/50 gamble, or its close substitute, a binary option. One can see from the expressions for the optimal trading strategies that they indeed contain binary option-type components. What if, perhaps more realistically, the manager is unable to synthetically create a binary option, as would be the case, for example, in the popular two-period model with continuous state space but with a finite number of securities available for investment? The above argument indicates that in such an economy the manager would clearly benefit from introduction of specific securities into her investment opportunity set: binary options. We note that this discussion applies generally to any preferences exhibiting a nonconcavity, and is not driven by the fact that in our setting the manager's payoff essentially includes a binary option.

Appendix C: Numerical Procedure

For the cases where an analytical solution is not available, we solve the model numerically. Our numerical solution utilizes a Monte Carlo simulation (e.g., Boyle, Broadie, and Glasserman (1997)). The simulation presented here is for the economic setting of Section 2, with $Y_t = S_t$. We first simulate the relative portfolio value $\hat{V}_t(\pi_t)$ using equation (A8). This simulation requires knowledge of the distribution of π_T under the G measure (proof of Proposition 2), which in our setup is

$$\pi_T = \pi_t \, e^{[-(1-\gamma)(\mu-\gamma\sigma^2/2)-(\gamma\sigma-\kappa)^2/2](T-t)+(\gamma\sigma-\kappa)\sqrt{T-t}\,z},$$

where z is a standard normal random variable. The second step is to compute the trading strategy financing $\hat{V}_t(\pi_t)$. Following standard arguments, we obtain the optimal risk exposure expression as

$$\hat{\theta}_t(\pi_t) = 1 + \frac{\partial \hat{V}_t(\pi_t)}{\partial \pi_t} \frac{\pi_t}{\hat{V}_t} \left(\gamma - \frac{\kappa}{\sigma}\right). \tag{A12}$$

The derivative $\frac{\partial \hat{V}_t(\pi_t)}{\partial \pi_t}$ is computed numerically from the simulated $\hat{V}_t(\pi_t)$ schedule.

Briefly, the details of the procedure are as follows.

(a) Fix π_t. Simulate M normal variates, $z^1, \ldots, z^M$, and for each normal variate z^i compute

$$F^i_{t,T} \equiv e^{[-(1-\gamma)(\mu-\gamma\sigma^2/2)-1/2(\gamma\sigma-\kappa)^2](T-t)+(\gamma\sigma-\kappa)\sqrt{T-t}\,z^i} \quad \text{and} \quad \pi^i_T = \pi_t\, F^i_{t,T}.$$

(b) Determine the relevant critical values of π_T. This step depends on the form of the fund flow function. Use the concavification points $\underline{V}$ and $\overline{V}$ (Appendix B) to determine the value of π_T at which the risk-shifting range occurs. For example, for the collar-type fund flow with Condition 1 violated, compute

$$\pi^* = u'_L(\underline{V}) = u'_M(\underline{V}).$$

(c) Determine V^i_T for a given π^i_T. This step again depends on the flow specification. Continuing with the collar-type fund flow with Condition 1 violated, we compute the optimal V^i_T for each segment of $u(\cdot)$ (Appendix B), as well as its value over the risk-shifting range. Compute V^i_T from the following formulas:

$$V^i_T = \begin{cases} \frac{1}{f_H}\left(\frac{\pi^i_T}{f_H}\right)^{-1/\gamma} & \text{if} \quad \pi^i_T < \pi_{**}, \\ e^{\eta_H} & \text{if} \quad \pi_{**} \le \pi^i_T < \pi_*, \\ \text{Solve } u'_M(V_T) = \pi^i_T \text{ numerically for } V_T & \text{if} \quad \pi_* \le \pi^i_T < \pi^*, \\ \frac{1}{f_L}\left(\frac{\pi^i_T}{f_L}\right)^{-1/\gamma} & \text{if} \quad \pi^i_T \ge \pi^*, \end{cases}$$

where $\pi_* = u'_M(e^{\eta_H})$ and $\pi_{**} = u'_H(e^{\eta_H})$. The flat segment over (π_{**}, π_*) is due to the kink in the utility function upon transitioning from $u_M(\cdot)$ to $u_H(\cdot)$, and the discontinuity at π^* is due to the risk-shifting range. The initial portfolio value is chosen such that the Lagrange multiplier on the budget constraint $\hat{y}$ is unity.

(d) Compute $\hat{V}(\pi_t) = \frac{1}{M}\sum_{i=1}^M \rho_t F^i_{t,T} V^i_T$.

(e) Repeat the procedure (steps (a)-(d)) for a range of π_t, step size $\Delta\pi$. Save the array $\hat{V}(\pi_t)$.

(f) Compute the trading strategy using (A12), approximating the derivative $\frac{\partial \hat{V}_t(\pi_t)}{\partial \pi_t}$ by the finite difference $\frac{\partial \hat{V}_t(\pi_t)}{\partial \pi_t} \approx \frac{\hat{V}(\pi_t+\Delta\pi)-\hat{V}(\pi_t)}{\Delta\pi}$.

(g) Plot $\theta_t(\pi_t)$ as a function of $\log \hat{V}(\pi_t)$ parametrically. This is the desired optimal risk exposure, depicted in Figure 3.

Table 1a
Costs and benefits of active management in economies (a)

The investor's gain/loss quantified in units of his initial wealth, $\hat{\lambda}$, solves $V^I((1+\hat{\lambda})W_0) = \hat{V}(W_0)$, where $V^I(\cdot)$ denotes the investor's indirect utility under his optimal policy θ^I, and $\hat{V}(\cdot)$ his indirect utility under delegation. The gain due to explicit incentives, λ^N, solves $V^I((1+\lambda^N)W_0) = \hat{V}(W_0;\, f_T = 1)$, where $\hat{V}(W_0;\, f_T{=}1)$ denotes the investor's indirect utility under delegation absent implicit incentives. The gain due to implicit incentives, λ^Y, solves $1+\hat{\lambda} = (1+\lambda^N)(1+\lambda^Y)$. The fixed parameter values are (where applicable) $\gamma = 1$, $\gamma_I = 2$, $f_L = 0.8$, $f_H = 1.5$, $f_L + f_H = 2.3$, $\beta = 1$, $\eta_L = -0.08$, $\eta_H = 0.08$, $\eta_L + \eta_H = 0$, $\mu = 0.08$, $r = 0.02$, $\sigma = 0.16$, $W_0 = 1$, $T = 1$.

Effects of		Cost-benefit measures λ^Y, λ^N $\hat{\lambda}$ (%)				
Managerial risk	γ	0.5	1.0	1.5	2.0	2.25
aversion		-28.86, -27.12	-6.68, -3.45	-5.05, -0.39	-3.45, 0.00	-2.80, -0.04
		-48.16	-9.90	-5.42	-3.45	-2.83
Implicit reward	f_H-f_L	0.3	0.5	0.7	0.9	1.1
for outperformance		-2.82, -3.45	-4.80, -3.45	-6.68, -3.45	-8.48, -3.45	-10.26, -3.45
		-6.17	-8.09	-9.90	-11.64	-13.36
Risk exposure	θ^Y	0.50	0.75	1.00	1.25	1.50
of the benchmark		-3.99, -3.45	-5.00, -3.45	-6.68, -3.45	-8.97, -3.45	-11.79, -3.45
		-7.31	-8.28	-9.90	-12.12	-14.83
Flow threshold	η_H-η_L	0.08	0.12	0.16	0.20	0.24
differential		-2.06, -3.45	-4.49, -3.45	-6.68, -3.45	-8.65, -3.45	-10.42, -3.45
		-5.44	-7.79	-9.90	-11.81	-13.52

Table 1b
Costs and benefits of active management in economies (b)

The investor's gain/loss quantified in units of his initial wealth, $\hat{\lambda}$, solves $V^I((1+\hat{\lambda})W_0) = \hat{V}(W_0)$, where $V^I(\cdot)$ denotes the investor's indirect utility under his optimal policy θ^I, and $\hat{V}(\cdot)$ his indirect utility under delegation. The gain due to explicit incentives, λ^N, solves $V^I((1+\lambda^N)W_0) = \hat{V}(W_0;\, f_T = 1)$, where $\hat{V}(W_0;\, f_T{=}1)$ denotes the investor's indirect utility under delegation absent implicit incentives. The gain due to implicit incentives, λ^Y, solves $1+\hat{\lambda} = (1+\lambda^N)(1+\lambda^Y)$. The fixed parameter values are (where applicable) $\gamma = 1$, $\gamma_I = 2$, $f_L = 0.8$, $f_H = 1.5$, $f_L + f_H = 2.3$, $\beta = 1$, $\eta_L = -0.08$, $\eta_H = 0.08$, $\eta_L + \eta_H = 0$, $\mu = 0.06$, $r = 0.02$, $\sigma = 0.29$, $W_0 = 1$, $T = 1$.

Effects of		Cost-benefit measures λ^Y, λ^N $\hat{\lambda}$ (%)				
Managerial risk	γ	0.5	1.0	2.0	3.0	4.0
aversion		-8.13, -4.19	-5.12, -0.47	-3.31, 0.00	-2.56, -0.05	-2.15, -0.11
		-11.98	-5.61	-3.31	-2.61	-2.27
Implicit reward	f_H-f_L	0.3	0.5	0.7	0.9	1.1
for outperformance		-3.33, -0.47	-4.29, -0.47	-5.12, -0.47	-6.01, -0.47	-6.88, -0.47
		-3.79	-4.74	-5.61	-6.46	-7.32
Risk exposure	θ^Y	0.50	0.75	1.00	1.25	1.50
of the benchmark		-5.43, -0.47	-4.63, -0.47	-5.12, -0.47	-6.69, -0.47	-8.45, -0.47
		-5.88	-5.08	-5.61	-7.13	-8.88
Flow threshold	η_H-η_L	0.08	0.12	0.16	0.20	0.24
differential		-4.33, -0.47	-4.70, -0.47	-5.12, -0.47	-5.67, -0.47	-6.21, -0.47
		-4.78	-5.15	-5.61	-6.12	-6.65

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