Lifetime Consumption and Investment for Retirement

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Life-cycle Finance

- extremely important
- neglected in the academic literature
- primitive in practice
- institutions are complex
- quantitatively challenging

My contribution to this area is to build quantitative models of the individual choice problem. These models may not be rich enough to apply directly in practice, but they give us important economic insights. In a complex environment, we may not be able to write down or solve a fully realistic choice problem, but our heuristics can still be guided by intuition that is informed by theory and knowledge of the data.

Underlying Research

The main economic results have been published in the *Journal of Economic Theory* and a second paper with more on the mathematics will go to an applied mathematics journal.

Dybvig, Philip H., and Hong Liu, 2010, "Lifetime consumption and investment: Retirement and constrained borrowing," *Journal of Economic Theory* **145**, 885–907.

Dybvig, Philip H. and Hong Liu, 2009, "Verification Theorems for Models of Optimal Consumption and Investment with Retirement and Constrained Borrowing," Washington University working paper, available at

http://phildybvig.com/papers/DybvigLiuVerification.pdf.

Goals of the research

We build a workhorse model of lifetime consumption and investment for answering questions about retirement, pensions, and insurance. Some features of the model include:

- voluntary or mandatory retirement date
- nonnegative wealth constraint on borrowing
- idealized hedging using life insurance
- bequest motive possible
- can handle wage and mortality varying through life
- wages vary randomly over time

Main results

- Flexibility in retirement implies (nonrandom wage case)
  - working longer when the economy is bad
  - human capital has a negative beta
  - young people would like to hedge negative beta with a long stock position
- a borrowing constraint limits hedging opportunities and tempers the portfolio choice
- risky human capital reduces or reverses all of these effects
- discontinuity of portfolio choice at retirement
- closed form solutions

Specialized assumptions for today's presentation

Today's presentation focuses on a simple version of the model in which much of the solution can be computed directly. This version is easier to understand than the first example we worked out, which required relatively more numerical work and included stochastic wages with age-dependent average wage and mortality profiles (more on this later).

- constant wage (over time and states of nature)
- constant mortality rate
- retirement is either
  - mandatory (fixed)
  - voluntary
- insurance is fairly priced[1] and continuously available
- no bequest

[1] In the notation of the paper, $\iota = \delta$

Some notation

- c_t – consumption at time t
- $u(c_t) = \frac{{c_t}^{1-\gamma}}{1-\gamma}$ – felicity of consumption if not retired
- $u(Kc_t)$ – felicity of consumption if retired, $K > 1$
- δ – hazard rate for mortality (constant for this version of the model)
- ρ – pure rate of time discount
- r – riskfree rate
- $dS_t/S_t = \mu dt + \sigma dB_t$ – stock price process
- $\kappa \equiv (\mu - r)/\sigma$ – price of risk
- θ_t – holding in the stock (wealth units)
- τ – voluntary retirement date
- T – mandatory retirement date
- $k < 1$ – retirement income fraction
- W^* – initial wealth

Choice problem: voluntary retirement

Choose the retirement date τ (a stopping time), the adapted consumption process c_t, and the adapted portfolio choice θ_t to maximize

$$E\left[\int_{t=0}^{\tau} e^{-(\rho+\delta)t}u(c_t)dt + \int_{t=\tau}^{\infty} e^{-(\rho+\delta)t}u(Kc_t)dt\right]$$

subject to the budget conditions on the wealth process

$$W_0 = W^*$$

$$dW_t = \begin{cases} (r+\delta)W_t dt + \theta_t(\mu - r)dt + \theta_t\sigma dB_t - c_t dt + w dt & \text{for } t \leq \tau \\ (r+\delta)W_t dt + \theta_t(\mu - r)dt + \theta_t\sigma dB_t - c_t dt + kw dt & \text{for } t \geq \tau \end{cases}$$

$$(\forall t \geq 0)W_t \geq 0$$

standard except voluntary retirement date τ, mortality δ, preference for not working K, and post-retirement income multiplier k

Choice problem: mandatory retirement

Given the mandatory fixed retirement date T, choose the adapted consumption process c_t, and the adapted portfolio choice θ_t to maximize

$$E\left[\int_{t=0}^{T} e^{-(\rho+\delta)t}u(c_t)dt + \int_{t=T}^{\infty} e^{-(\rho+\delta)t}u(Kc_t)dt\right]$$

subject to the budget conditions on the wealth process

$$W_0 = W^*$$

$$dW_t = \begin{cases} (r+\delta)W_t dt + \theta_t(\mu - r)dt + \theta_t\sigma dB_t - c_t dt + w dt & \text{for } t \leq T \\ (r+\delta)W_t dt + \theta_t(\mu - r)dt + \theta_t\sigma dB_t - c_t dt + kw dt & \text{for } t \geq T \end{cases}$$

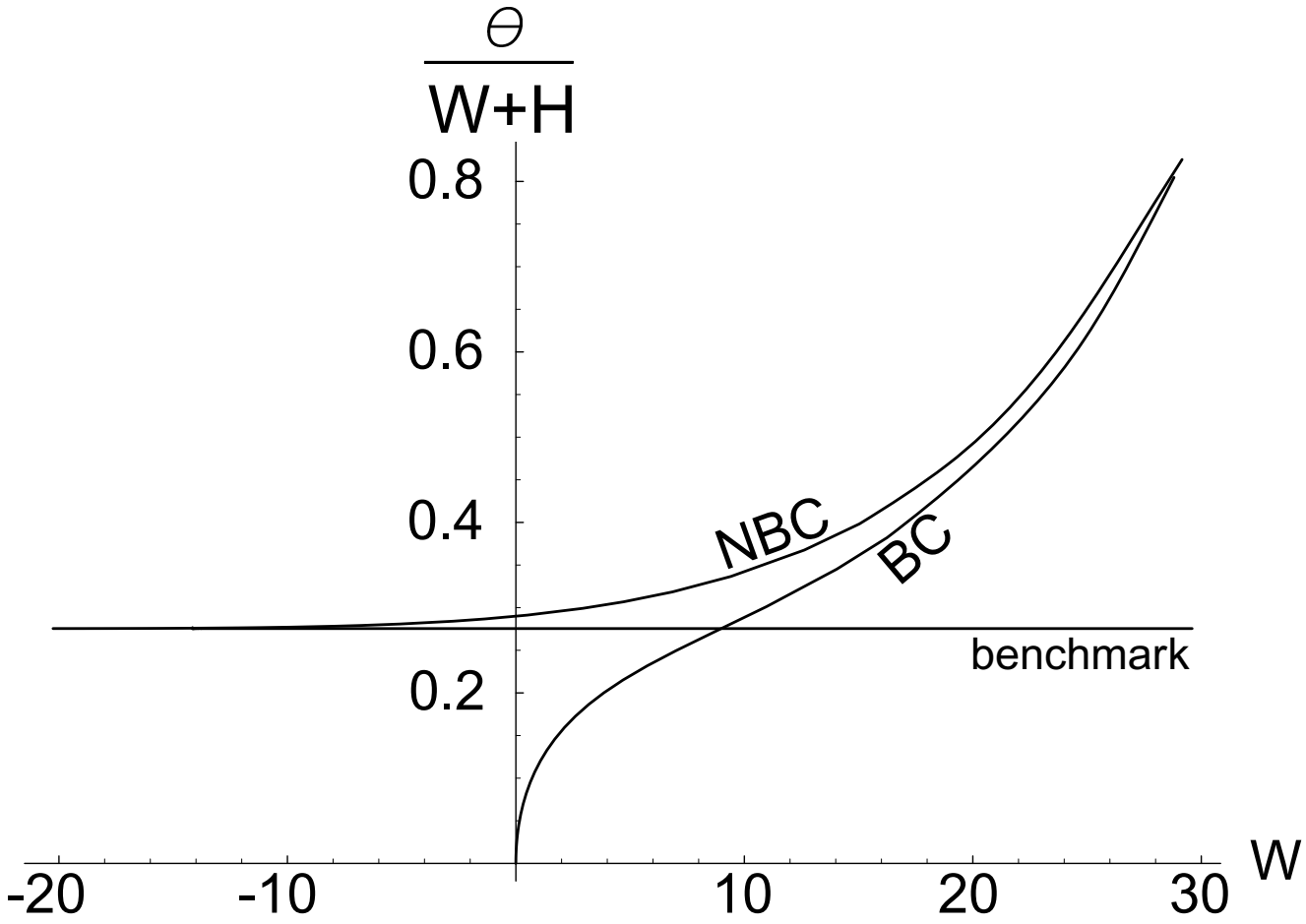
$$(\forall t \geq 0)W_t \geq 0$$

standard except mandatory retirement date T, mortality δ, preference for not working K, and post-retirement income multiplier k

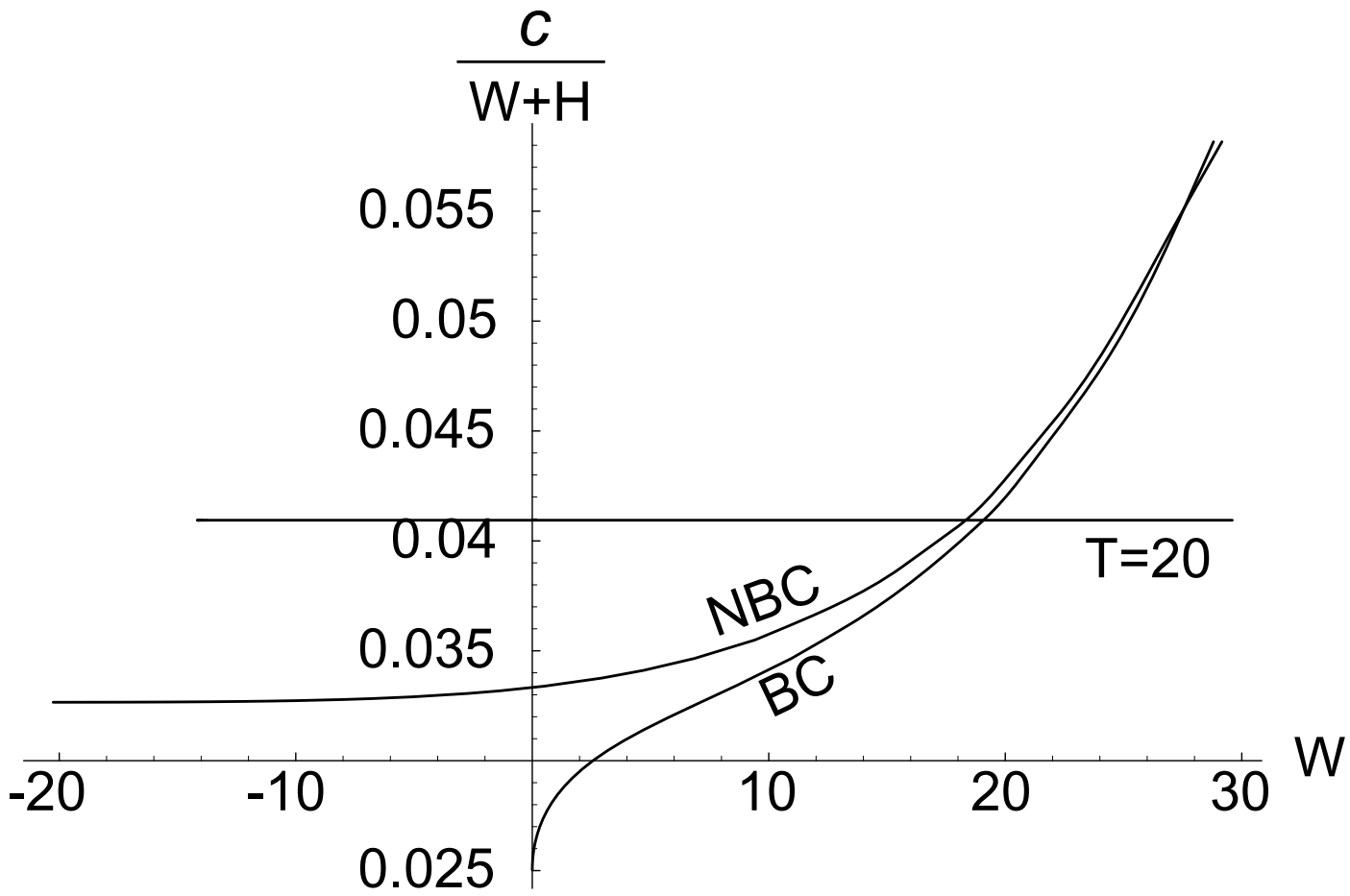
Solution technique: all cases use the dual approach

- using a change of variables to marginal utility (or marginal felicity)
- can be viewed as a solution to a convex dual to the original problem
- as in He in Pagès [1993] but with some new twists
- gives a linear differential equation more generally than the one-shot approach of Pliska [1982,1986] and others including Cox and Huang [1989]
- post-retirement, we have the same explicit solution to both problems
- pre-retirement solution for voluntary retirement
  - explicit solution up to one constant
- pre-retirement solution for mandatory retirement
  - in the dual, approximate the fixed maturity by a sequence of random regimes
  - Erlang distribution, a sum of Poissons, stationary each stage
  - from Liu and Loewenstein [2002] and Carr [1998]
  - recursive solution: solve for one constant in each stage

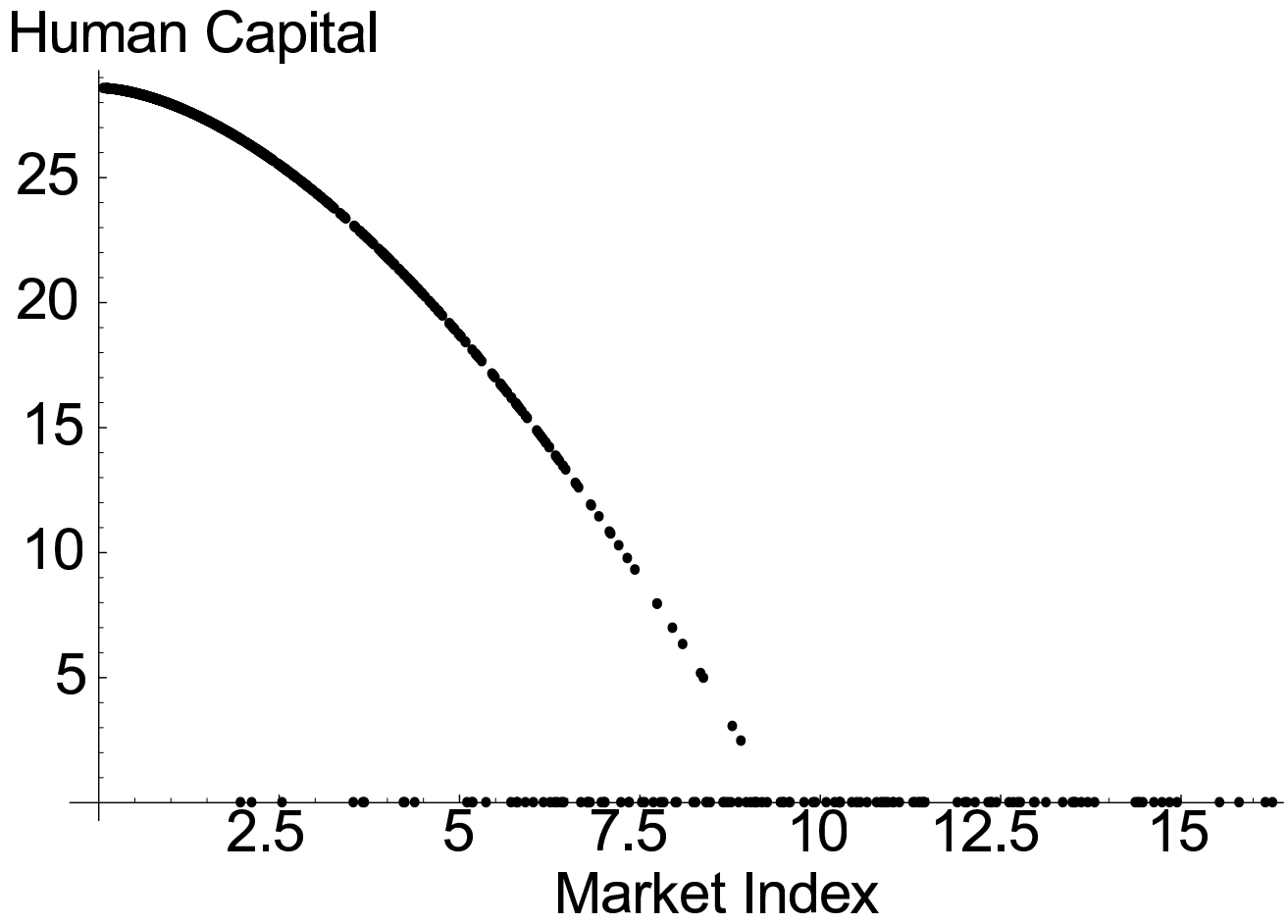
Equity per total wealth against financial wealth



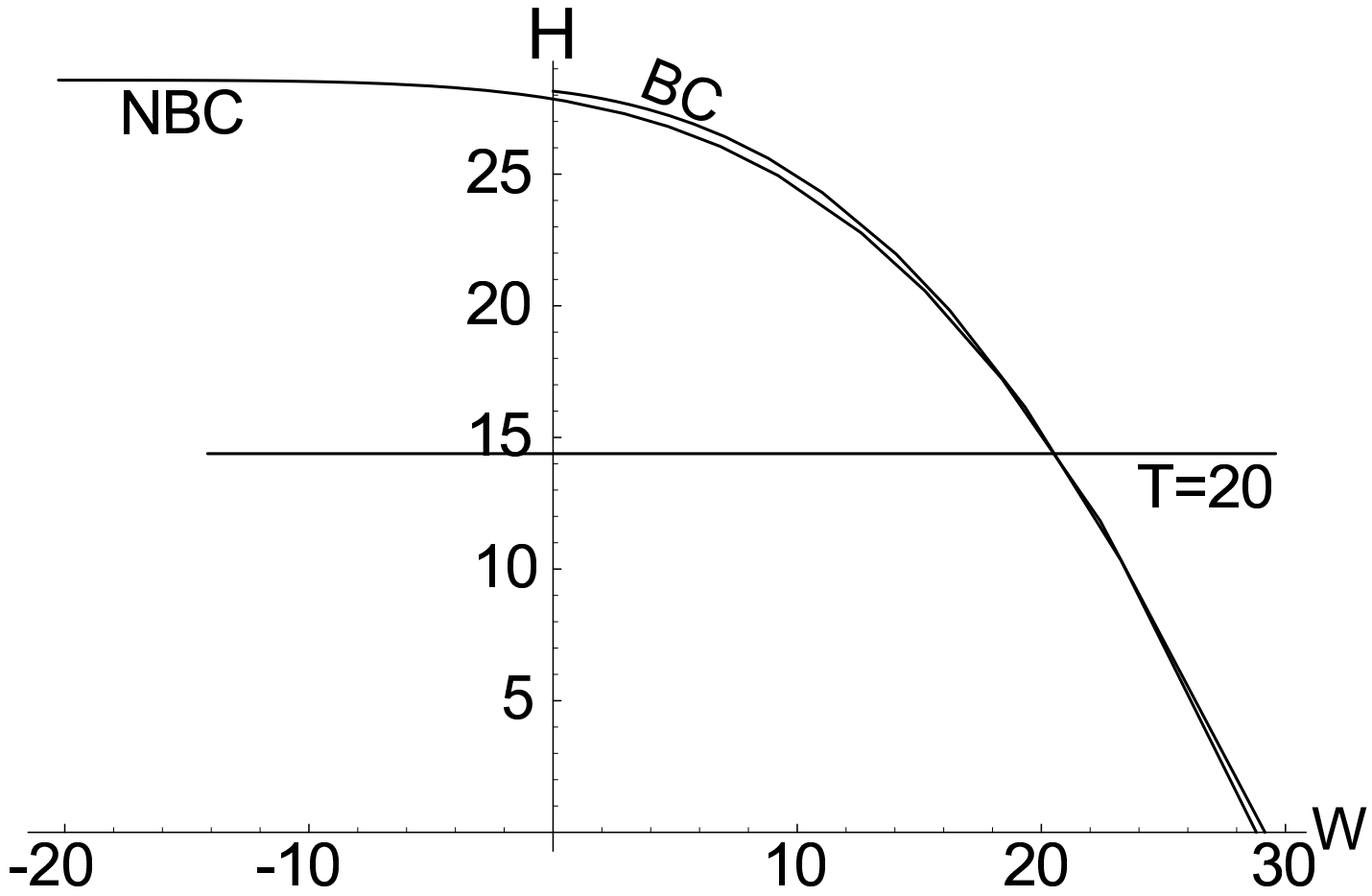
Consumption per total wealth against financial wealth



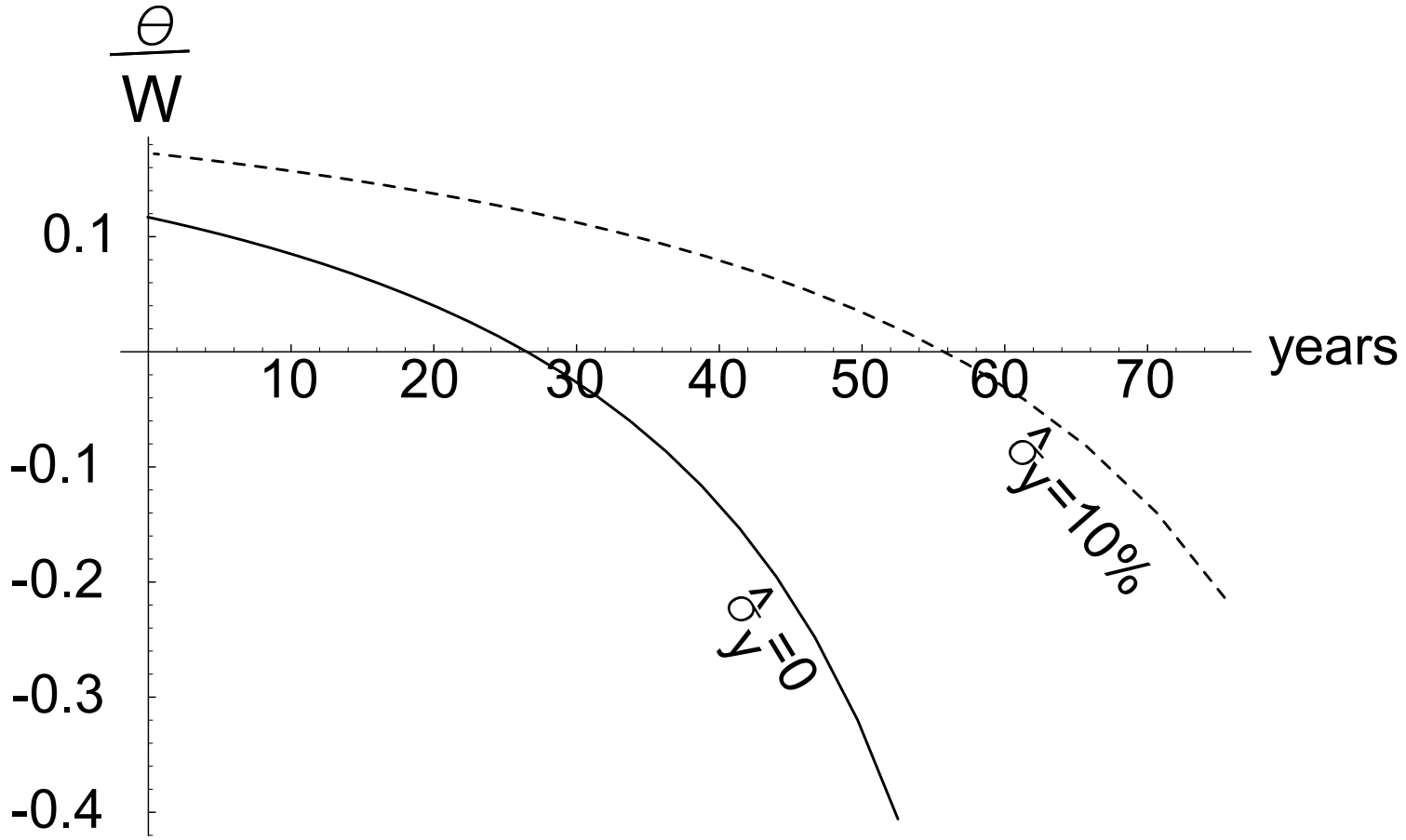
Human capital against the market index



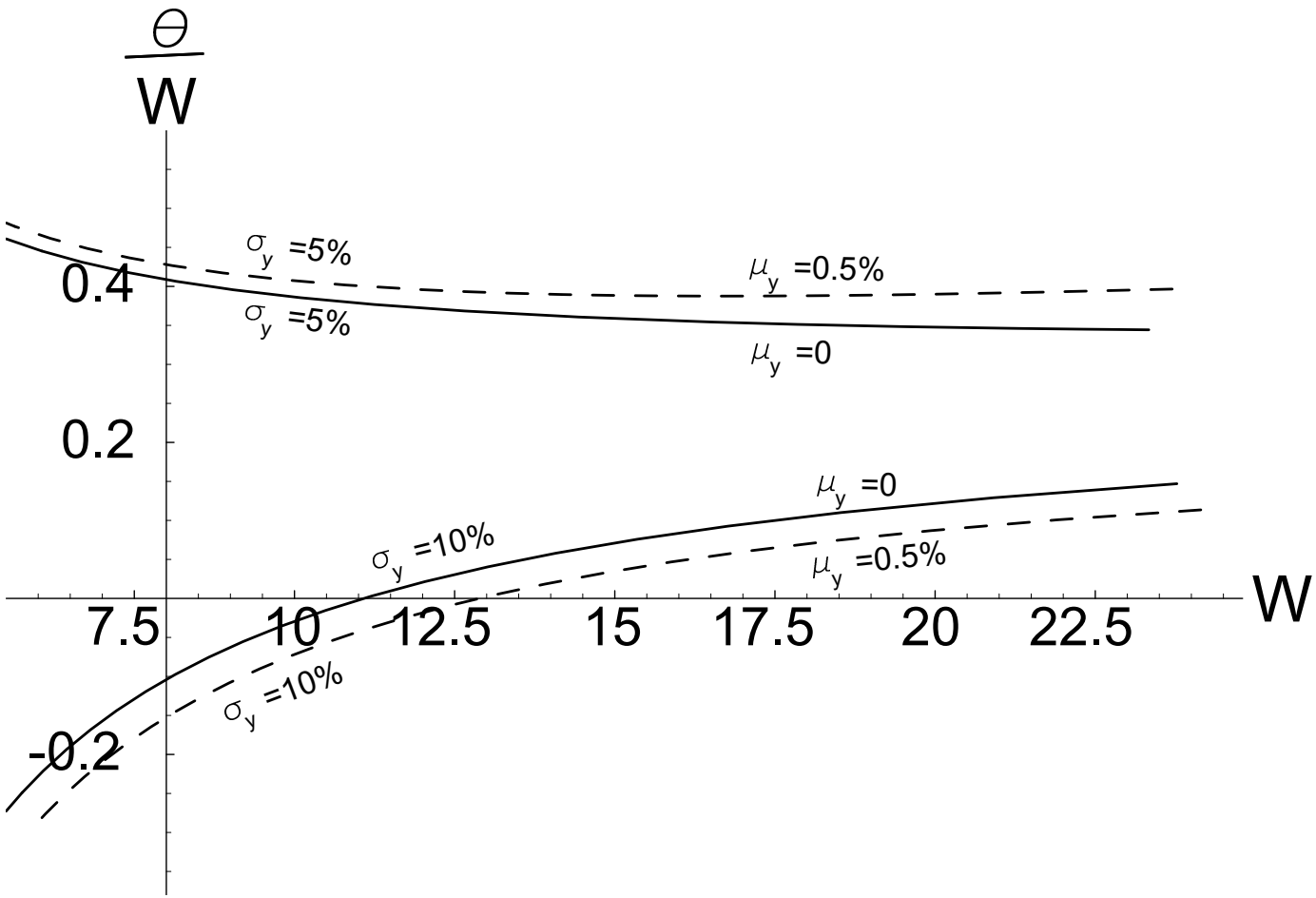
Human capital against financial wealth

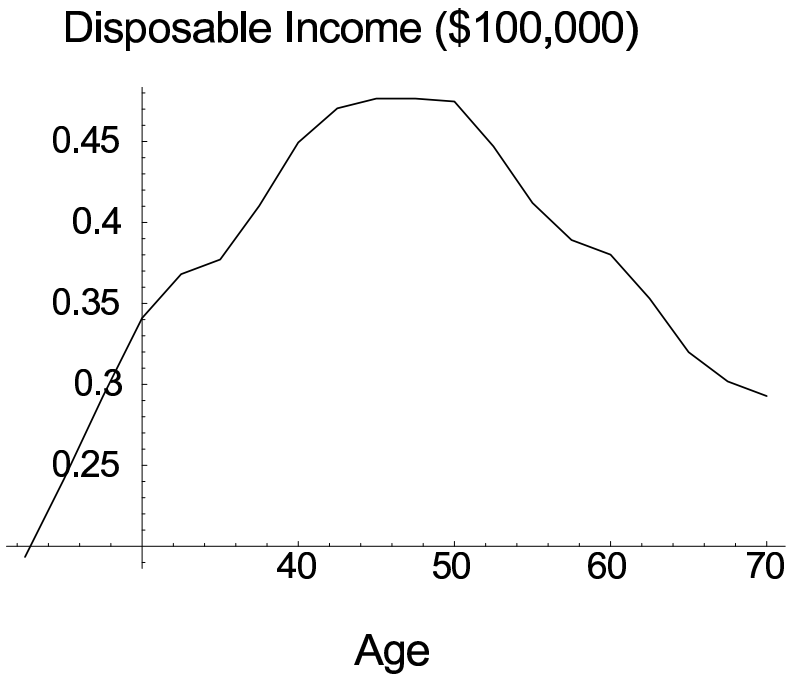


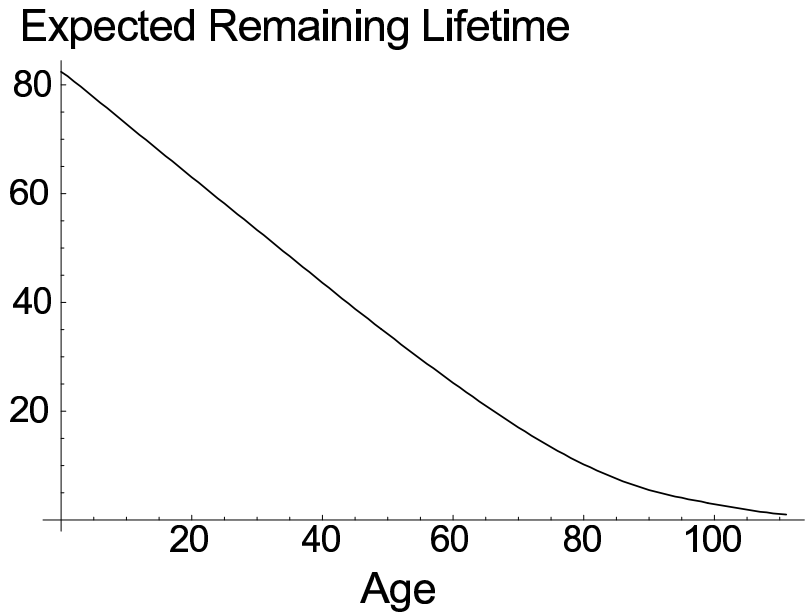
Market-sensitive labor: equity fraction and expected time to retirement



Idiosyncratic risk moderates the hedging demand.

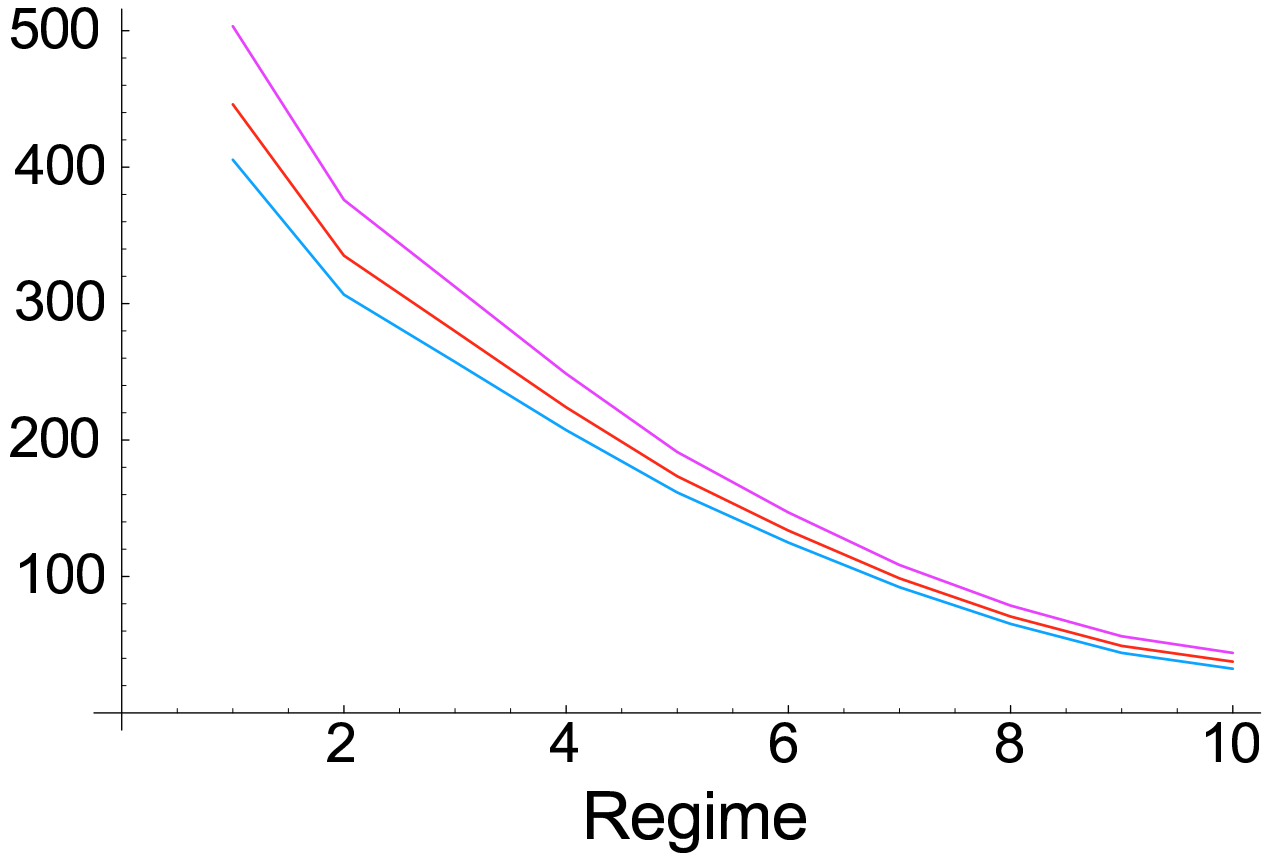

Θ/W
σy =5%
μy =0.5%
σy =5%
μy =0
0.4
0.2
μy =0
σy =10%
μy =0.5%
W
7.5
10
12.5
15
17.5
20
22.5
σy =10%
-0.2





Our original analysis assumed a lifetime income and mortality profile similar to empirical data.

Wealth threshold for retirement



Retirement is age dependent in the original analysis. We think of this as an intermediate case between the stationary voluntary retirement case in most of these slides (which has a horizontal boundary) and the mandatory retirement case (which has a vertical boundary).

Conclusion

We have constructed a workhorse model for analyzing retirement, pensions, and life insurance. The model is able to accomodate:

- voluntary or mandatory retirement date
- nonnegative wealth constraint on borrowing
- idealized hedging using life insurance
- bequest motive or not
- can handle wage and mortality varying through life
- stochastic wages over time