Stock market diversity

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Abstract

Stock market diversity is a measure of the distribution of capital in an equity market. Diversity is higher when capital is more evenly distributed among the stocks in the market, and is lower when capital is more concentrated into a few of the largest companies. This article reviews the measurement of diversity, the behavior of diversity over time, and the effect of diversity on portfolio performance.

1 Introduction

Stock market diversity, first considered in Fernholz (1999), is a measure of the distribution of capital in an equity market. Diversity is higher when capital is more evenly distributed among the stocks in the market, and is lower when capital is more concentrated into a few of the largest companies. There exist many measures of diversity, of which entropy is perhaps the best known, but not necessarily the most useful for our purposes. Market diversity, as measured by any of the measures of diversity, appears to be mean-reverting over the long term with intermediate-term trends.

Certain measures of diversity *generate* portfolios, generically called *diversity-weighted* portfolios, and these portfolios have a more even distribution of capital than the market. The relative return of a diversity-weighted portfolio is perfectly correlated with the change in market diversity as determined by the measure that generates it. Certain diversity-weighted portfolios can be shown to have a higher return than the market portfolio, with about the same level of risk, at least over the long term.

It appears that active equity managers as a group hold portfolios with a distribution of capital that is closer to a diversity-weighted portfolio than to the market. This may enhance the managers' returns over the long term, but it also causes their short-term relative returns to be correlated to changes in market diversity. There are a number of ways to measure the effect of changes in market diversity on portfolio performance, but it appears that the commonly-used least-squares regression techniques are likely to underestimate this effect.

In the next section we present some basic concepts from stochastic portfolio theory that we shall need later on. In the subsequent sections we introduce measures of diversity, we consider portfolios generated by measures of diversity, and we study the effect of changes in market diversity on portfolio performance. Mathematical proofs will not be included here, but can be found in Fernholz (2002), along with a more comprehensive treatment of the theory in general.

2 Stochastic portfolio theory

Consider a market of n stocks represented by *stock price processes* $X_1, \ldots, X_n$ that satisfy the stochastic differential equations

$$d \log X_i(t) = \gamma_i(t)\, dt + \sum_{\nu=1}^{n} \xi_{i\nu}(t)\, dW_\nu(t), \quad t \in [0, \infty), \tag{1}$$

for $i = 1, \ldots, n$, where $(W_1, \ldots, W_n)$ is (multivariate) Brownian motion, γ_i and $\xi_{i\nu}$, $i, \nu = 1, \ldots, n$ are measurable, adapted (which means that they do not depend on future events), and satisfy certain regularity conditions (see Fernholz (2002), Definition 1.1.1). A stochastic process of the form (1) is called a *continuous semimartingale*, and such processes are discussed in detail in Karatzas and Shreve (1991). The value $X_i(t)$ represents the price of the ith stock at time t, and we shall assume that there is a single share of stock outstanding for each company, so $X_i(t)$ represents the total capitalization of the ith company at time t. In (1), $d \log X_i(t)$ represents the *log-return* (or *continuous return*) of X_i over the (short) time period dt. The process γ_i in (1) is called the *growth rate process* for X_i, and the process $\xi_{i\nu}$ measures the sensitivity of X_i to the νth source of uncertainty, W_ν.

The *covariance process* for X_i and X_j is given by

$$\sigma_{ij}(t) = \sum_{\nu=1}^{n} \xi_{i\nu}(t)\xi_{j\nu}(t), \quad t \in [0, \infty),$$

with the notation $\sigma_i^2(t) = \sigma_{ii}(t)$ for the *variance* processes. It is commonly assumed that the matrix $(\sigma_{ij}(t))$ is strongly nonsingular in the sense that all its eigenvalues are bounded away from zero for all $t \geq 0$, and we shall do so here. It is not difficult to add dividend processes to our model, but for simplicity we shall assume here that stocks do not pay dividends.

Equation (1) is the *logarithmic representation* of the stock price processes, and this representation is further developed in Section 1.1 of Fernholz (2002). The logarithmic representation is quite general, and is equivalent to the usual arithmetic representation commonly used in mathematical finance (see, e.g., Karatzas and Shreve (1998)). The use of the logarithmic representation makes no assumption that we wish to maximize logarithmic utility, or, for that matter, any other utility function. We use the logarithmic representation because it brings to light certain aspects of portfolio behavior that remain obscure with the conventional arithmetic representation.

In the logarithmic representation we consider the growth rate γ_i, which can be interpreted as the expected rate of change of the logarithm of the stock price at time t. In the arithmetic representation, the *rate of return* α_i, rather than the growth rate γ_i, is considered for each stock X_i. The relation between these two variables is

$$\alpha_i(t) = \gamma_i(t) + \frac{\sigma_i^2(t)}{2}, \quad t \in [0, \infty) \quad \text{a.s.},$$

and this follows from Itô's rule (see Karatzas and Shreve (1991)), and is discussed in Section 1.1 of Fernholz (2002). Let us now consider portfolios, and their growth rates and variances.

We represent a portfolio π by its *weight processes* $\pi_1, \ldots, \pi_n$ where $\pi_i(t)$ is the proportion of the portfolio invested in X_i at time t. The weight processes are assumed to be adapted and bounded, and must sum to one, so we have $\pi_1(t) + \cdots + \pi_n(t) = 1$, a.s., for all t. A negative value of $\pi_i(t)$ indicates a short sale in X_i.

If we let $Z_\pi(t)$ represent the value of π at time t, then Z_π will satisfy the stochastic differential equation

$$\frac{dZ_\pi(t)}{Z_\pi(t)} = \sum_{i=1}^n \pi_i(t)\frac{dX_i(t)}{X_i(t)}, \quad t \in [0,\infty). \tag{2}$$

This equation shows that the instantaneous return on the portfolio π is the weighted average of the instantaneous returns of the stocks in the portfolio.

It can be shown (by application of Itô's rule, see Fernholz (2002), Corollary 1.1.6) that (2) can be expressed in logarithmic form as

$$d\log Z_\pi(t) = \sum_{i=1}^n \pi_i(t)\, d\log X_i(t) + \gamma_\pi^*(t)\, dt, \quad t \in [0,\infty), \quad \text{a.s.}, \tag{3}$$

where

$$\gamma_\pi^*(t) = \frac{1}{2}\Big(\sum_{i=1}^n \pi_i(t)\sigma_{ii}(t) - \sum_{i,j=1}^n \pi_i(t)\pi_j(t)\sigma_{ij}(t)\Big) \tag{4}$$

is called the *excess growth rate*. The last summation on (4) represents the portfolio variance rate for π, and this expression is identical to the portfolio variance rate in the arithmetic representation. It can be seen from (4) that the excess growth rate is one half the difference of the weighted average of the stock variances minus the portfolio variance. In this sense the excess growth rate measures the efficacy of diversification in reducing the portfolio risk.

However, the excess growth rate measures more than the reduction of portfolio risk. It can be shown that the excess growth rate $\gamma_\pi^*(t)$ is always positive for a portfolio with no short sales, at least if the portfolio holds at least two stocks (Fernholz (2002), Proposition 1.3.7). From (1) and (3), we see that the *portfolio growth rate* γ_π will satisfy

$$\gamma_\pi(t) = \sum_{i=1}^n \pi_i(t)\gamma_i(t) + \gamma_\pi^*(t), \quad t \in [0,\infty), \quad \text{a.s.},$$

so at a given time t, π will have a higher growth rate than the weighted average of the growth rates of its component stocks. Hence, superior diversification not only reduces risk, but also increases the growth rate of a portfolio.

Perhaps the most important portfolio we shall consider is the *market portfolio* μ with weights

$$\mu_i(t) = \frac{X_i(t)}{X_1(t) + \cdots + X_n(t)}, \quad t \in [0,\infty), \tag{5}$$

for $i = 1, \ldots n$. For this portfolio with appropriate initial conditions, the portfolio value is

$$Z_\mu(t) = X_1(t) + \cdots + X_n(t), \quad t \in [0,\infty), \quad \text{a.s.}, \tag{6}$$

which is the total capitalization of the market, as we would expect for the market portfolio. The weights $\mu_i(t)$ are called *capitalization weights* or *market weights*. From (5) and (6) we have

$$\log \mu_i(t) = \log X_i(t) - \log Z_\mu(t), \quad t \in [0,\infty), \quad \text{a.s.},$$

so

$$d\log \mu_i(t) = d\log X_i(t) - d\log Z_\mu(t), \quad t \in [0,\infty), \quad \text{a.s.}, \tag{7}$$

and we see that the logarithmic change in the market weight μ_i is equal to the log-return of X_i relative to the market. Using (3) and (7), we can express the log-return of a portfolio π relative to the market as

$$d\log\big(Z_\pi(t)/Z_\mu(t)\big) = \sum_{i=1}^n \pi_i(t)\, d\log \mu_i(t) + \gamma_\pi^*(t)\, dt, \quad t \in [0,\infty), \quad \text{a.s.}$$

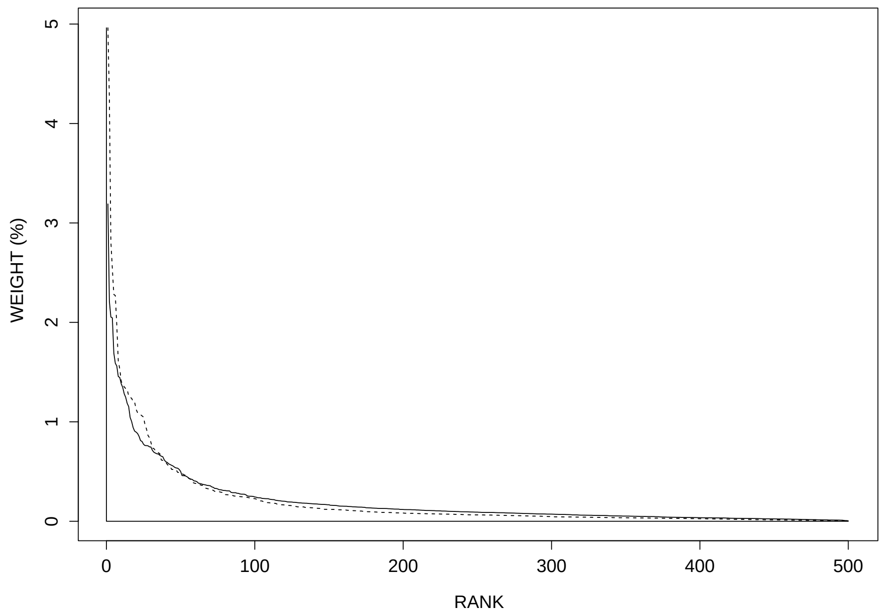

Figure 1: Capital distribution of the S&P 500 Index.
December 30, 1997 (solid line), and December 29, 1999 (broken line).

This equation shows that the relative log-return of π is the weighted average of the relative log-returns of the stocks in the portfolio, plus the excess growth rate γ_π^*.

3 Stock market diversity

The *capital distribution curve* of the market, or of a cap-weighted index, is the graph of the market weights arranged in decreasing order. Figure 1 shows the capital distribution curves for the S&P 500 Index on December 30, 1997 and December 29, 1999. As we see from the chart, there was a significant concentration of capital into the largest stocks in the market over this period. We shall now discuss methods by which the level of diversity of the market can be measured quantitatively. We shall also investigate the idea that a more even distribution of capital among the larger and smaller stocks is likely to improve portfolio performance.

Let $\mu(t) = (\mu_1(t), \ldots, \mu_n(t))$ represent the vector of market weights at time t. A *measure of diversity* is a function of the market weights that is positive, symmetric, and concave. The archetypal measure of diversity is perhaps the *entropy* function

$$S(\mu(t)) = -\sum_{i=1}^{n} \mu_i(t) \log \mu_i(t), \quad t \in [0, \infty),$$

introduced by Shannon (1948) when he invented information theory. Entropy can be used to measure market diversity, but other measures are more suited to our purposes. The function

$$\mathbf{D}_p(\mu(t)) = \Big(\sum_{i=1}^{n} \mu_i^p(t)\Big)^{1/p}, \quad t \in [0, \infty), \tag{8}$$

where $0 < p < 1$, is a measure of diversity that satisfies $1 < \mathbf{D}_p(\mu(t)) \leq n^{(1-p)/p}$ (see Fernholz

Figure 2: Cumulative change in market diversity, 1927–2004.

(2002)). Besides having a bounded logarithm, the measure $\mathbf{D}_p$ has the advantage that the parameter p can be adjusted to accommodate particular circumstances that may arise in practice.

In Figure 2 we see the cumulative changes in the diversity of the U.S. stock market over the period from 1927 to 1999, measured by $\mathbf{D}_p$ with $p = 1/2$. The chart shows the cumulative changes in diversity due to capital gains and losses, rather than absolute diversity, which is affected by changes in market composition and corporate actions. Considering only capital gains and losses has the same effect as adjusting the "divisor" of an equity index. The values used in Figure 2 have been normalized so that the average over the whole period is zero. We can observe from the chart that diversity appears to be mean-reverting over the long term, with intermediate trends of 10 to 20 years. The extreme lows for diversity seem to accompany bubbles: the Great Depression, the "nifty fifty" era, and the "irrational exuberance" period.

We say that a function $\mathbf{S}$ *generates* a portfolio π if

$$\log\big(Z_\pi(t)/Z_\mu(t)\big) = \log \mathbf{S}(\mu(t)) + \Theta(t), \quad t \in [0,\infty), \quad \text{a.s.}, \tag{9}$$

where the *drift process* Θ is of locally bounded variation. In this case, $\mathbf{S}$ is called the *generating function* of π, and π is called a *functionally generated portfolio* (see Fernholz (2002) for a development of these ideas). Since Θ is of locally bounded variation, and since processes of bounded variation have no correlation with other processes, it follows from (9) that the relative log-return of π has correlation one with the change in the logarithm of the generating function. Hence, the generating function "explains" all of the quadratic variation of the relative performance of a functionally generated portfolio.

It can be shown that if a function $\mathbf{S}$ is positive over the range of the market weights and twice continuously differentiable, then it generates a portfolio, so there is a good supply of functionally generated portfolios. Here we are interested in portfolios generated by measures of diversity, and it can be shown that these portfolios have a more even distribution of capital than the market. It can also be shown that these portfolios have increasing drift processes.

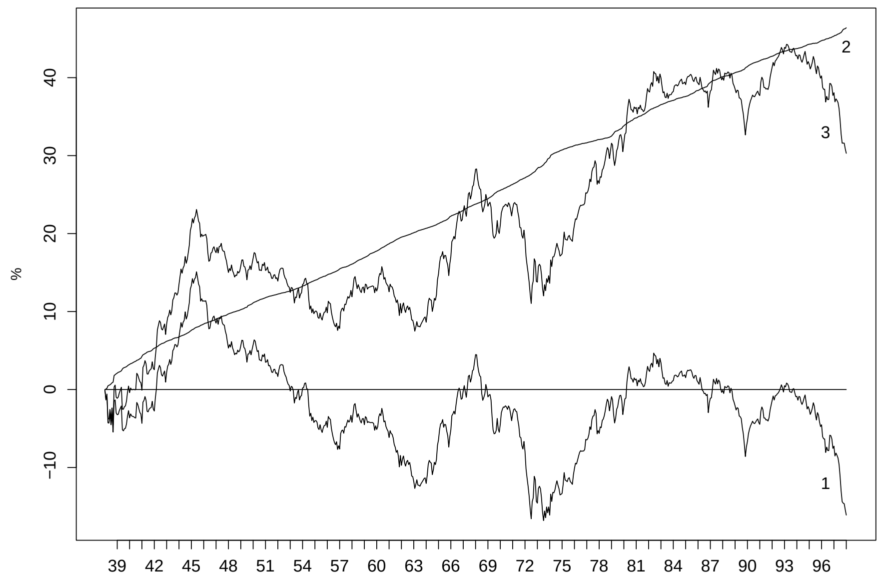

Figure 3: Simulation of a $\mathbf{D}_p$-weighted portfolio, 1939–1998
1: $\log \mathbf{D}_p(\mu(t))$; 2: $\Theta(t)$; 3: $\log\bigl(Z_\pi(t)/Z_\mu(t)\bigr)$

As an example, let us consider the portfolio π generated by $\mathbf{D}_p$. This portfolio has weights

$$\pi_i(t) = \frac{\mu_i^p(t)}{\bigl(\mathbf{D}_p(\mu(t))\bigr)^p}, \quad t \in [\,0, \infty), \tag{10}$$

for $i = 1, \ldots, n$, and (9), in differential form, becomes

$$d \log\bigl(Z_\pi(t)/Z_\mu(t)\bigr) = d \log \mathbf{D}_p(\mu(t)) + (1-p)\gamma_\pi^*(t)\,dt, \quad t \in [\,0, \infty), \quad \text{a.s.} \tag{11}$$

The portfolio π is called a $\mathbf{D}_p$-*weighted* or, generically, a *diversity-weighted* portfolio. Such portfolios are practical for equity management; in fact, a $\mathbf{D}_p$-weighted portfolio with $p = .76$ and composed of the stocks in the S&P 500 Index has been used for institutional investors in the U.S. (see Fernholz et al. (1998)). Since $0 < p < 1$, it can be seen from (10) that relative to the market, π is underweighted in the larger stocks and overweighted in the smaller stocks, and that it has an increasing drift function.

We have seen that $\log \mathbf{D}_p(\mu(t))$ is bounded, and since the last term in (11) is positive, it will eventually dominate the right-hand side of the equation. This argument has been used by Fernholz et al. (2005) to show that *relative arbitrage* exists in certain hypothetical markets, and it also shows that in actual markets, a bias toward the smaller stocks will augment relative portfolio performance over the long term. Moreover, this enhanced performance does not depend on higher risk for the smaller stocks since there is no assumption that smaller stocks have higher risk in the market model in (1). It is not difficult to see that for the U.S. market, over periods of the order of magnitude of the relaxation time of the time series in Figure 2, the risk must be about the same as that of the market. Hence, for investors with an investment horizon of 10 to 20 years, diversity-weighted portfolios would seem to have an advantage over cap-weighted indices.

To get some idea of the behavior of a diversity-weighted portfolio for actual stocks, we ran a simulation using the stock database from the Center for Research in Securities Prices (CRSP) at the University of Chicago. The data included 60 years of monthly values from 1939 to 1998 for exchange-traded stocks after the removal of closed-end funds, REITs, and ADRs not included in the S&P 500 Index. From this universe, we considered a cap-weighted large-stock index consisting of the largest 1000 stocks in the database for those months that the database contained 1250 or more stocks, and the largest 80% of the stocks those months that the database contained fewer than 1250. Against this index, we simulated the performance of the corresponding $\mathbf{D}_p$-weighted portfolio, with $p = 1/2$. No trading costs were included.

The results of the simulation are presented in Figure 3: Curve 1 is the change in the generating function, Curve 2 is the drift process, and Curve 3 is the relative return. Each curve shows the cumulative value of the monthly changes induced in the corresponding process by capital gains or losses in the stocks, so the curves are unaffected by monthly changes in the composition of the database. As can be seen, Curve 3 is the sum of Curves 1 and 2. The drift process $\Theta(t)$ was the dominant process over the period, and was remarkably stable, with a total contribution of 46.4 percentage points to the relative return. To calculate the total relative return of an investment in the $\mathbf{D}_p$-weighted index versus an investment in the cap-weighted index, dividend payments must also be considered. However the difference in dividend payments between the two portfolios was quite small, with a total contribution over the entire period of only 1.3% in favor of the cap-weighted index. (Figure 3 does not include dividend payments.)

4 Manager performance and the diversity cycle

Active equity managers strive to find portfolios that will outperform the market. Since we have seen that portfolios that are less concentrated into the largest stocks are likely to outperform the market at least over the long term, it might not be surprising to find that active equity managers have a similar weighting bias. If active managers have a bias toward more diverse portfolios, we would expect their short-term relative returns to depend to some extent on changes in the diversity of the market. Active managers would be expected to do better when diversity is increasing, and worse when diversity is decreasing. This was shown indeed to be the case by Fernholz and Garvy (1999).

In Figure 4 we have plotted the annual logarithmic return relative to the S&P 500 Index of the median manager from the Domestic Equity Database of Callan Associates for the years 1973 through 2003 versus the logarithmic change in $\mathbf{D}_p$ for the largest 1000 stocks in the CRSP universe of stocks. Each data point is represented in the chart by the year to which it pertains. The diagonal line is the least-squares regression line. Analysis of the regression indicates that more than half of the annual quadratic variation of the managers' relative log-returns is explained by the change in diversity ($R^2 = .58$). The slope of the regression line is not due to the "outliers" at 1998 and 2000; a robust regression procedure, least-trimmed-squares (see Marazzi (1993)), gives a slightly steeper slope. Hence, we find that change in diversity is an important variable in explaining relative manager performance.

Since the time series in Figure 2 appears to be mean-reverting with intermediate trends, it may be possible to forecast changes in diversity to some extent. If the performance of active managers depends on change in diversity, then such forecasts could be useful in asset allocation between active and passive equity management. During those periods where diversity appears to be rising, assets could be allocated to active managers; during periods of declining diversity, passive equity management would probably be preferable. Of course, Figure 4 represents the effect of diversity changes on active managers as a group; it is also important to be able to measure the effect of diversity change on a specific manager, and we shall consider this problem in the next section.

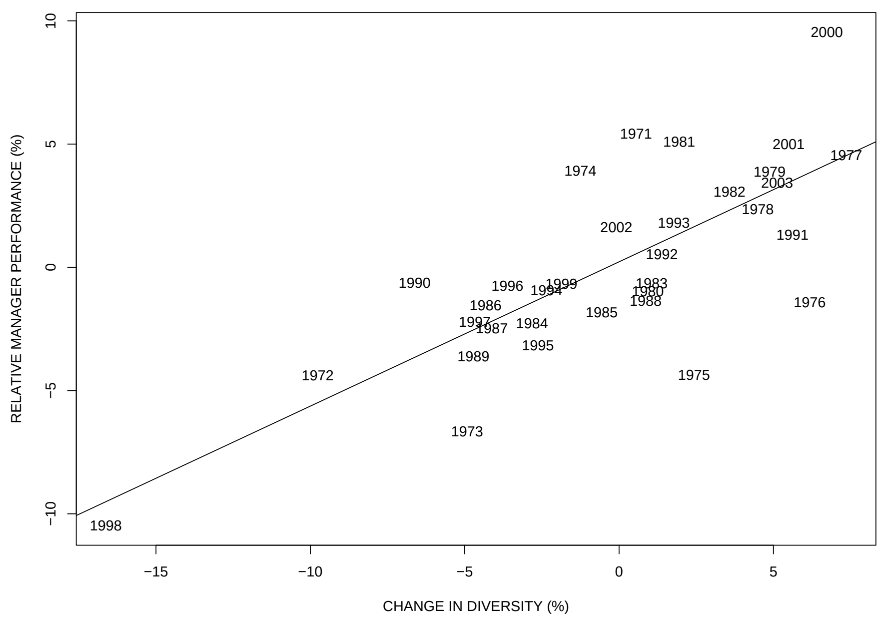

Figure 4: Annual relative manager performance vs. change in diversity.

5 The distributional component of equity return

A conventional method for measuring the effect of a change in diversity on the portfolio performance is to use least-squares regression with some “size factor” as the explanatory variable. However, there are several problems with this method, among them that the choice of size factor is rather arbitrary, and that least-squares regression carries a number of assumptions that do not pertain in reality. Here we propose an alternative methodology developed in Fernholz (2001).

In Figure 1 we observed the capital distribution curves for the S&P 500 Index on December 30, 1997 and December 29, 1999. As we see in the chart, there was a considerable concentration of capital into the largest stocks between these two dates. A change of this nature in the capital distribution curve is likely to have an effect on the relative return of a portfolio, and we would like to be able to measure what that effect is.

Figure 5 is a stylized version of the capital distribution curves in Figure 1. Suppose that the solid line represents the capital distribution at a given time t_0, and the broken line represents the distribution t_1 sometime later. Suppose a particular stock starts at position A at t_0 and ends up at position C at time t_1. In this case, the weight of this stock in the market (or index) has increased, so it has had a higher return than the market over the period $[t_0, t_1]$. However, it has fallen in rank: if it had maintained its initial rank, it would be at position B at time t_1. The return implied by a move from A to B is called the *distributional component* of the relative return of this stock over $[t_0, t_1]$. In this case, the distributional component of the relative return is greater than the relative return itself, so the difference, which is called the *residual return*, is negative.

By calculating the distributional components of the relative returns of all the stocks in a portfolio, and taking the weighted average of these, we can determine the distributional component of the relative return of an arbitrary portfolio. For a diversity-weighted portfolio, the distributional component is identical to the change in diversity over the period, so our direct measurement procedure can be considered to be a generalization of the generating-function decomposition in (11).

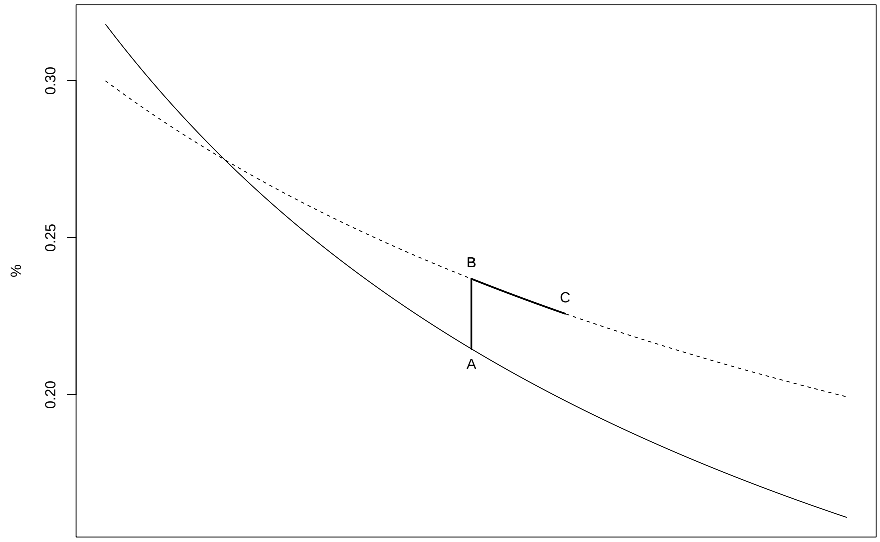

Figure 5: A–B is the distributional component of the stock return.

To test this method of direct measurement, we apply it to the a simulated portfolio of an "active-core" manager over the ten-year period from 1989 through 1998. We can see from Figure 2 that over this period market diversity declined substantially, so if our simulated manager shared the median manager's the propensity to overweight small stocks, this should have been detrimental to performance.

Figure 6 shows the simulated manager's log-return relative to the S&P 500 Index, and we see that the manager outperformed the index by about 10% over the first five years, but over the second five years there was actually slight underperformance. What happened? In Figure 7 we see the cumulative distributional component of the manager's return, calculated by the direct method (solid line). The declining curve indicates that the change in the capital distribution over the period caused a significant problem for this manager.

Suppose now that we had used standard least-squares regression to analyze the simulated manager's performance. First, since there is no canonical time series to represent "size", we must arbitrarily choose such a series: let us use the relative return of the largest 100 stocks in the S&P 500 for our size series. In Figure 7, the broken line represents the estimate of the size component of the manager's relative return using least-squares regression. The regression estimate shows only about a fifth of the cumulative effect shown by the direct measurement. It may be of interest to note that a robust regression procedure, least trimmed squares (see Marazzi (1993)), gives estimates quite close to our direct measurement.

6 Conclusion

Market diversity appears to be an important factor in equity portfolio performance. Holding a portfolio that is closer to diversity weights than to capitalization weights should improve relative performance over the long term. Change in diversity appears to be highly correlated with the relative performance of managers, and since change in diversity may be amenable to statistical prediction, this could have significant implications for asset allocation between active and passive equity managers. Direct measurement shows a greater effect on portfolio return due to changes in the capital distribution than measurement by least-squares regression.

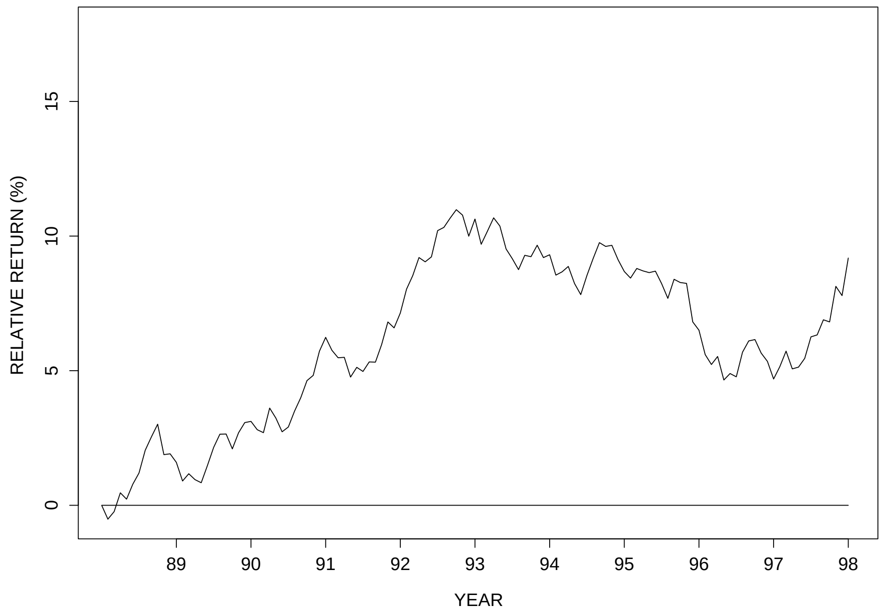

Figure 6: Relative return of a simulated manager.

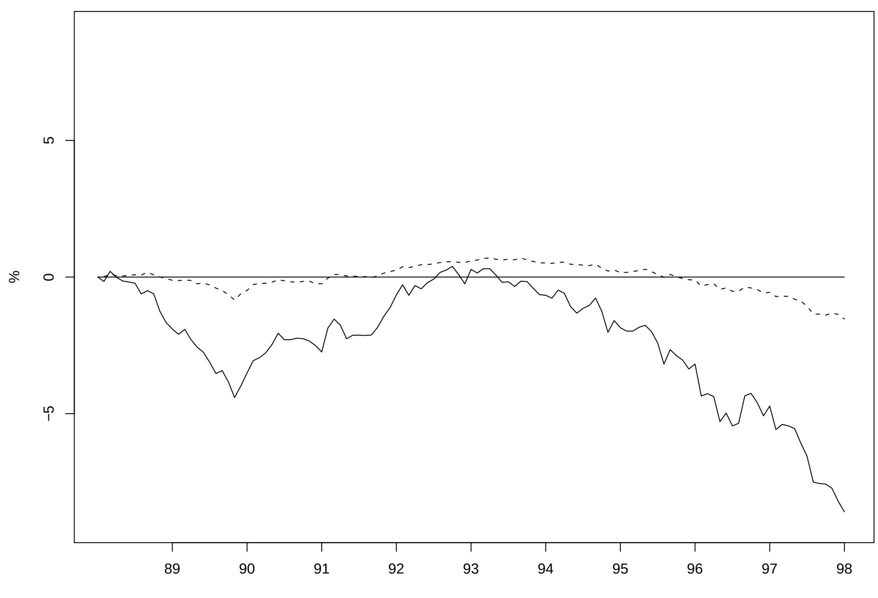

Figure 7: Cumulative distributional component for the simulated manager.
Solid line: distributional component by direct measurement.
Broken line: size component by regression with the top 100 stocks.

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