

# Optimal Portfolio Choice over the Life Cycle with Social Security

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## Abstract

This paper examines how households should optimally allocate their portfolio choices between risky stocks and risk-free bonds over their lifetime. Previous lifecycle models suggest that the allocation toward stocks should start high early in life and decline over a person's age as human capital depreciates. Moreover, these models suggest that the share of savings devoted to stocks should be roughly independent of full lifetime income in the presence of homothetic utility.

However, the actual empirical evidence indicates more of a “hump” shape allocation: the share of savings devoted toward stocks starts low early in life, peaks in midlife, and declines toward retirement. Moreover, the lifetime poor hold a smaller percentage of their portfolio in stocks relative to higher-income groups.

We reconsider the portfolio choice allocation in a computationally-demanding lifecycle model in which households face uninsurable wage shocks and an uncertain lifetime. Quite critically, we include a progressive social security system that is *wage indexed* before retirement, as in many pay-as-you-go systems including the United States. Social security benefits, therefore, are correlated with stock returns at the low frequency that is relevant for lifecycle retirement planning. We show that this model is able to more closely replicate the key stylized facts of portfolio choice.

## 1 Introduction

The investment choice of savings is among the most important decisions over the life cycle. Not surprisingly, a considerable literature has examined how households should optimally allocate their savings between stocks and bonds over their lifecycle. Samuelson (1969) and Merton (1969) first addressed this problem in discrete and continuous time, respectively, assuming complete markets without labor income. Merton (1971) then introduced deterministic labor income into the continuous time framework while Bodie, Merton and Samuelson (1992) allowed for it to be elastically supplied. In more recent years, attention

has shifted toward examining how uninsurable labor income risk and uncertain lifespans impact savings and portfolio allocations over the life cycle.<sup>1</sup>

It is well known, however, that the standard expected-utility model does not appear to closely match the empirical evidence on portfolio choice.<sup>2</sup> The actual empirical evidence suggests three “stylized facts:” (I) the share of a household’s portfolio invested in equities is much less than 100% for most households; (II) the lifetime poor invest in fewer equities than richer households; and (III) the share of portfolio invested in risky assets tend to be “hump shape” ( $\cap$ ) in age. (Amerkis and Zeldes 2000; Heaton and Lucas 2000; Poterba and Samwick, 2002). In contrast, the standard model with homothetic preferences tends to generate a very high stock allocation across all income classes that declines with age as human capital depreciates; in many portfolio choice models, the allocation is counterfactually “U shaped” in age.

Several modifications to the standard model have been proposed in the past to deal with at least one of these empirical observations: liquidity constraints (Brown 1990; Amerkis and Zeldes 2000); saving for illiquid assets such as a house (Faig and Shum 2002); habit persistence (Polkovnichenko, 2007); and, incomplete trading markets between generations in a real business cycle economy where wages and stock returns are perfectly correlated (Storesletten, Telmer, and Yaron, 2007). In contrast, the current paper preserves the standard life-cycle expected utility model with homothetic utility but simply considers the importance of a more detailed fiscal setting: a progressive and wage-indexed social security system.

In the United States, a retiree’s initial Social Security benefit level is calculated by first multiplying each previous wage earned by the participant with a wage-indexed factor. This factor tracks the average wage growth experienced across the entire economy since the time that particular wage was earned. For example, suppose that you retired today and you earned \$30,000 twenty years ago. Also suppose that wages, averaged across the entire economy, grew on average of 5% (nominal) per year during the past 20 years. Your recorded wage of \$30,000 would, therefore, be adjusted upward by a factor of  $1.05^{20} = 2.65$ , or equal \$79,598.93. A similarly constructed wage factor would be applied to each of your other previous wages as well. Your individual Average Indexed Yearly Earnings (AIYE) would then be determined by taking the average of your adjusted wages.<sup>3</sup> Your AIYE is then used to calculate your Primary Insur-

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<sup>1</sup>See Bertaut and Haliassos (1997); Gakidis (1998); Storesletten, Telmer, and Yaron (2000, 2007); Viceira (2001); Bodie, Detemple, and Walter (2004); Benzoni, Collin-Dufresne and Goldstein (2004); Cocco, Gomes and Maenhout (2005) and Gomes and Michaelides (2005). Related papers by Balduzzi and Lynch (1997), Brennan and Xia (1998), Campbell and Viceira (1999), and Kim and Omberg (1996) study the effects caused by the changes of riskfree interest rate or equity premium over the time. Carroll (2002) and Campbell and Cochrane (1999) propose models in which risk aversion varies with wealth.

<sup>2</sup>Of course, a plausible model should also match key stylized facts regarding consumption profiles and wealth-income ratios over the lifecycle, which we discuss later. Previous models, however, have been more successful at generating these features.

<sup>3</sup>Technically speaking, in the United States, this calculation is done at monthly (not yearly) frequency to calculate the Average Indexed Monthly Earnings (AIME). Moreover, only the best “best 35” years of earnings – those with the largest adjusted wages – are included in

ance Amount (PIA) using a nonlinear (progressive) function. Your PIA forms the basis of your Social Security benefit amount before some additional adjustments are made (e.g., a spousal benefit). Individuals with a larger AIYE receive a larger PIA, thereby recognizing that they paid more into the system. However, the PIA / AIYE ratio – the “replacement rate” – is a declining function of the AIYE so that the lifetime poor receives a relatively larger replacement rate of their previous earnings.

Empirically, the Social Security wage index factors, which are used to adjust previous earnings, are highly correlated with stock returns. At first glance, this fact might be surprising since it is well known that average wages and stock returns are *not* highly correlated at “high” (annual) frequency even at the sectorial level (e.g., Davis and Paul 2000). However, there are two distinct differences that creates a higher correlation between the Social Security wage index factors and stock returns. First, “idiosyncratic” (including sector-specific) risks tend to cancel inside of the *economy-wide average* index wage factor that is used to adjust previous wages for calculating Social Security benefits. Second, and more importantly, this indexation is mostly calculated at a “low frequency,” especially for previous wages earned early in the lifecycle. The economy-wide average wage and stock returns appear to be much more highly correlated at this lower frequency (Jermann, 1999).<sup>4</sup>

Wage indexation has presumably not been considered in the previous literature due to the additional computational complexity it brings by adding an additional state variable. We show, however, that a wage-indexed social security system can potentially play an important role in reconciling all three stylized facts noted above. First, a wage-indexed social security system provides a retirement asset that is highly correlated with stock returns, thereby reducing the demand for stocks by most households (stylized fact (I)). Second, this effect is especially strong for lower-income households who derive most of their retirement income from social security due to its progressive benefit structure (stylize fact (II)). For poor households (many of whom don’t have access to employer-based defined-contribution retirement plans), we also investigate how a realistic minimum investment account requirement in stocks reinforces the pattern observed in our model. Third, the correlation between social security and equity returns is larger at *younger* ages, reducing the demand for equity. As benefits begin to accrue at higher ages and the horizon toward retirement becomes shorter, social security benefits begin to become more substitutable with bonds, thereby increasing the demand for equities. Closer to retirement, enough human capital has depreciated, making stocks less attractive again (stylized fact (III)).

One might object to our positive findings by arguing that few people actually understand the complexities of how their social security benefits are calculated. We tend to agree. However, our results could also be interpreted from a nor-

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the AIME. We ignore these finer details in order to keep the state space more manageable. The U.S. system also has a cap on yearly wages subject to the payroll tax, and the AIME is, therefore, calculated over capped wages. Our calculations include this cap.

<sup>4</sup>This result is consistent with a neoclassical model with shocks to productivity and depreciation.

mative angle as well. In particular, from an empirical perspective, the average household might be already holding something close to the optimal portfolio. As a result, very little change is needed in their behavior to make it optimal.

The organization of the paper is as follows. Section 2 present a general overview of the empirical evidence from the Survey of Consumer Finances. Section 3 sets up the model and discusses the assumption and numerical methods. Section 4 lays out the calibration of the parameters. Section 5 presents the benchmark simulation results. Section 6 conducts some sensitivity analysis. Section 7 concludes.

## 2 Empirical Evidence

The Survey of Consumer Finances (SCF) is generally considered the best data source for reporting the financial wealth of U.S. families. It provides comprehensive coverage of different measures of wealth as well as key characteristics (age, education, etc.) that tend to be correlated with wealth. The SCF is conducted every three years with different families; we report evidence from the most recent sample from the year 2004.<sup>5</sup> However, as Amerkis and Zeldes (2000) warn, the exact magnitude of these estimates should be taken with a “grain of salt” due to data definitions and other issues. Appendix A describes the definitions and variables from SCF 2004 that we use in our empirical estimation.

Figure 1 reports the average (mean) percent of a household’s portfolio invested in risky stocks by age for families in which the “household head” (primary earner) falls into one of three different education groups: No High School education; with High School education only; with College education. Households are grouped into 5-year “age buckets” in order to increase the sample size per cell.

Figure 1 reveals the three key stylized facts that motivate this paper: (i) the share of a household’s portfolio invested in equities is much less than 100% for most households; (ii) the lifetime poor (those with less education) invest in fewer equities than richer households; and (iii) the share of portfolio invested in risky assets tend to be “hump shaped” (or “*inverse U*”) in age. For example, the

Figure 2 reports the average of total wealth relative to total income for these same education groups, respectively. Overall, wealth-income ratios increase with age and education. The age-shape of the accumulation profiles also suggest a significant bequest motive that is increasing in the education class. However, the exact steepness should also be taken with a “grain of salt” due to potential survivor bias where wealthier households tend to live longer on average.

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<sup>5</sup>The SCF is a non-panel data dataset that uses a dual-frame sample design. One part is selected to obtain a sufficiently large and unbiased sample. The other part is designed to disproportionately select wealthier families. We use the provided weights to adjust the unequal probabilities of selection in the survey and for nonresponse.

### 3 Specification of the Model

This section presents a lifecycle portfolio choice model with standard preferences where households face uncertain and uninsurable wage earnings as well as an uncertain length of life.<sup>6</sup> Individuals make their consumption, saving and portfolio decisions over risk-free bonds and risky stocks at the beginning of each period during their working years. Their savings and portfolio choices are motivated by three main factors: to buffer uncertainty in the presence of incomplete insurance markets (precautionary savings); consumption needs after retirement; and, a joy-of-giving desire to leave a bequest. Individuals also pay a social security tax during their working years in exchange for a social security benefit during retirement that augments their private savings.

#### 3.1 Preferences

Each individual  $i$  works from age  $t_0$  to retirement age  $M$ . The maximum length of life is  $T$  and  $p_t$  is the probability that the individual is still alive at age  $t + 1$  conditional on being alive at age  $t$ . Preferences take the standard Constant Relative Risk Aversion (CRRA) form and are additively separable over time. The household's maximum expected utility at  $t_0$ , therefore, is

$$\max E_{t_0} \sum_{t=t_0}^T \beta^{t-t_0} \left( \prod_{j=t_0-1}^{t-2} p_j \right) \left( p_{t-1} \frac{C_{i,t}^{1-\gamma}}{1-\gamma} + b(1-p_{t-1}) \frac{W_{i,t}^{1-\gamma}}{1-\gamma} \right) \quad (1)$$

where  $\beta < 1$  is the weight placed on future utility;  $C_{i,t}$  is  $i$ 's consumption at age  $t$ ;  $\gamma$  is the coefficient of relative risk aversion.  $W_{i,t}$  is the wealth (bequest) left at the time of the death where  $b$  identifies the intensity of the bequest motive.

#### 3.2 Financial Assets

Investors have access to two investment instruments: risk-free bonds and risky stocks. The riskless bonds pay a constant gross real return  $R_f$ , while the risky asset pays a gross real return  $R_t^S$ :

$$R_t^S = R^f + \mu + \eta_t \quad (2)$$

where  $\mu$  is the deterministic equity premium and  $\eta_t$  is the innovation to this excess returns of time  $t$ .  $\eta_t$  is independently and identically distributed (i.i.d.) over time and normally distributed  $N(0, \sigma_\eta)$ .

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<sup>6</sup>While wages are typically uninsurable in the private sector, private annuities directly hedge against an uncertain length of life, although these instruments are uncommonly used for this purpose (rather than tax benefits). In our baseline simulations, we assume that households do not annuitize, although our model could be easily solved allowing for perfect annuities.

### 3.3 Labor Income

Following Carroll (1997), Gourinchas and Parker (2002), Cocco, Gomes and Maenhout (2005), and several other papers, exogenous labor income for agent  $i$  at time  $t$  is modeled as

$$Y_{i,t} = \exp(y_{i,t}) \quad (3)$$

Before retirement, log income is composed of three terms:

$$y_{i,t} = g(t, F_{i,t}) + \omega_{i,t} + z_{i,t} \quad (4)$$

where  $g(t, F_{i,t})$  is a deterministic function of age  $t$  and other individualistic characteristics  $F_{i,t}$ ;  $\omega_{i,t}$  is an (idiosyncratic) i.i.d. normally distributed shock  $N(0, \sigma_\omega)$ ; and,  $z_{i,t}$  is a first-order (permanent) autoregressive shock

$$z_{i,t} = \theta z_{i,t-1} + \xi_t + \phi_{i,t} \quad (5)$$

with an autocorrelation coefficient equal to  $\theta$ ; and, an aggregate (economy-wide) component  $\xi_t$  and idiosyncratic component  $\phi_{i,t}$  that are distributed i.i.d. as  $N(0, \sigma_\xi)$  and  $N(0, \sigma_\phi)$ , respectively. We also allow the correlation between the innovation to excess stock returns  $\eta_t$  and the aggregate labor income shock  $\xi_t$  to equal  $\rho_{\xi\eta}$  so that  $\eta_t$  and  $\xi_t$  are jointly bivariate distributed (see the Appendix). Assuming rational expectations, while individuals don't know their income and stock returns in the next period, they are aware of the underlying distributions.

Individuals pay a flat social security tax rate of  $\tau_{SS}$  through their working years and a nonlinear labor income tax rate  $\tau_w$  calibrated below. The (net of tax) disposable labor income at time  $t$  for individual  $i$ , therefore, is:

$$\begin{aligned} Y_{i,t}^d &= (1 - \tau_{SS})(1 - \tau_w [Y_{i,t}])Y_{i,t} \text{ for } t < M \\ &= 0 \text{ for } t \geq M \end{aligned} \quad (6)$$

### 3.4 Wage-Indexed Social Security

Social security benefits received at the point of retirement at age  $M$  are based, in part, on the individual's Average Indexed Yearly Earnings (AIYE) during their working years. The AIYE is calculated by adjusting upward each wage earned by individual  $i$  during the previous year  $t$  by the growth in the aggregate (economy-wide) average wage between year  $t$  and the retirement year,  $M$ . Denote  $Y_t^A$  as the aggregate average wage level at time  $t$ . The appropriate wage index adjustment factor for wages earned in year  $t$ , therefore, equals  $\frac{Y_M^A}{Y_t^A}$ , which is the cumulative growth in aggregate wages between times  $t$  and  $M$ . Individual  $i$ 's AIYE at the point of retirement, therefore, is equal to the sum of all previous wages adjusted upward by their respective wage index factors, divided by the number of working years:<sup>7</sup>

<sup>7</sup>Technically, in the United States, only the "best 35 years" are actually counted in the computation of the average wage. Tracking that decomposition, however, would require several more state variables, and would not materially alter our conclusions.

$$\begin{aligned}\bar{Y}_{i,M} &= \frac{Y_{i,t_0} \frac{Y_M^A}{Y_{t_0}^A} + Y_{i,t_0+1} \frac{Y_M^A}{Y_{t_0+1}^A} + \dots + Y_{i,M-1} \frac{Y_M^A}{Y_{M-1}^A}}{M - t_0} \\ &= \frac{\left( \left( Y_{i,t_0} \frac{Y_{t_0+1}^A}{Y_{t_0}^A} + Y_{i,t_0+1} \right) \frac{Y_{t_0+2}^A}{Y_{t_0+1}^A} + Y_{i,t_0+2} \right) \dots}{M - t_0}\end{aligned}\quad (7)$$

The aggregate wage level is modeled as a first-order autoregressive process with a unit-root in logs:

$$Y_{t+1}^A = Y_t^A \cdot \exp \xi_{t+1} \quad (8)$$

which we can rewrite as:

$$\frac{Y_{t+1}^A}{Y_t^A} = \exp \xi_{t+1} \quad (9)$$

We can then track individual  $i$ 's AIYE at each age based on his AIYE in the previous age:<sup>8</sup>

$$\bar{Y}_{i,t+1} = \frac{(t - t_0) \bar{Y}_{i,t} \cdot \exp \xi_{t+1} + Y_{i,t+1}}{t - t_0 + 1} \quad (10)$$

Social security benefits at retirement, therefore, are a function of individual  $i$ 's AIYE:

$$\begin{aligned}SS_{i,t} &= Q(\bar{Y}_{i,M}) \text{ for } t \geq M \\ &= 0 \text{ for } t < M\end{aligned}$$

where  $Q()$  is a nonlinear function calibrated to the U.S. Social Security system (discussed below).

The correlation between the *aggregate* average wage and stock returns, therefore, is  $\rho_{\xi\eta} = \frac{cov(\xi_t, \eta_t)}{\sigma_\xi \cdot \sigma_\eta}$ . However, the correlation between *individual*  $i$ 's (log) wage and stock returns is approximately one-fourth of the size due to additional idiosyncratic risk that is present in the individuals wage:

$$\rho_{y\eta} = \frac{cov(\xi_t, \eta_t)}{\sigma_\eta \cdot \sqrt{\sigma_\xi^2 + \sigma_\phi^2 + \sigma_\omega^2}} \approx \frac{cov(\xi_t, \eta_t)}{\sigma_\eta \cdot \sqrt{16 \cdot \sigma_\omega^2}} = \frac{cov(\xi_t, \eta_t)}{4 \cdot \sigma_\xi \cdot \sigma_\eta}$$

The distinction between these correlations allows us to “dial up” the aggregate average correlation (without idiosyncratic terms) to a large value at generational-level frequency while preserving a much smaller correlation value that households individually face at high frequency.<sup>9</sup>

<sup>8</sup>This “rolling average” approach allows us to track the individual’s AWIE with the addition of just one state variable.

<sup>9</sup>To be sure, the aggregate average wage correlation with stocks will also be large even at a *high* frequency under this approach. (Distinguishing explicitly between low and high frequency values for the aggregate correlation would require additional state variables.) However, individual households never observe this particular correlation. Instead, they only see the high-frequency small correlation with their own wages and the low-frequency large correlation through eventual wage indexation of their future social security benefits.

### 3.5 Optimization Problem

As in several previous papers, we treat the share of income spent on housing,  $h_t$ , as an “above-the-line” expense that exogenously reduces pre-retirement disposable income. In each period  $t$ , individual  $i$  has “cash on hand” denoted as  $X_{i,t}$ , which is composed of the following resources: wealth  $W_{i,t}$  at the beginning of each period (which is equal to the bequest in the case of death); *plus*, disposable income net of housing expenditures received before retirement; *plus*, social security benefits (after retirement):

$$X_{i,t} = W_{i,t} + (1 - h_t)Y_{i,t}^d + SS_{i,t} \quad (11)$$

Given the cash on hand at time  $t$ , individual  $i$  then jointly decides how much to consume  $C_{i,t}$  and the share of the residual savings,  $X_{i,t} - C_{i,t}$ , that is invested into risky equities versus risk-free bonds. Denote  $\alpha_{i,t}$  as the proportion invested into the risky asset; hence,  $1 - \alpha_{i,t}$  is the share invested into risk-free bonds. We assume  $\alpha_{i,t} \in [0, 1]$ , so that the allocation into bonds and stocks has to be positive at any period; in our actual simulations, these constraints are not binding.<sup>10</sup>

For calibration purposes, we allow the investment return to be taxed at a rate  $\tau_d$ .  $R_{i,t}$  denotes the total gross real return on the portfolio at time  $t$ . When the net stock return is positive  $R_{i,t}^S > 1$ , the investor is taxed on both asset classes; otherwise, only the risk-free asset is taxed:<sup>11</sup>

$$\begin{aligned} R_{i,t} &= [\alpha_{i,t} \cdot R_{i,t}^S + (1 - \alpha_{i,t}) \cdot R_f - 1] \cdot (1 - \tau_d) + 1 \quad \text{for } R_{i,t}^S > 1 \\ &= \alpha_{i,t} \cdot R_{i,t}^S + (1 - \alpha_{i,t}) \cdot [(R_f - 1) \cdot (1 - \tau_d) + 1] \quad \text{for } R_{i,t}^S \leq 1 \end{aligned} \quad (12)$$

The amount of wealth the following period,  $t + 1$ , therefore, equals

$$W_{i,t+1} = R_{i,t}(X_{i,t} - C_{i,t}) \quad (13)$$

The model, therefore, has four state variables,  $\{t, X_{i,t}, z_{i,t}, \bar{Y}_{i,t+1}\}$ , along with two time-vectored control variables:  $\{C_{i,t}, \alpha_{i,t}\}_{t=t_0}^T$ .<sup>12</sup> In some of our simulations, we also require the total amount of saving invested into the risky asset,  $\alpha_{i,t}(X_{i,t} - C_{i,t})$  to exceed a minimum threshold  $m$  in each period of life. This

<sup>10</sup>Intuitively, a short position in equities ( $\alpha < 0$ ) would expose agent  $i$  to potentially unbounded losses, a risk that would never be taken under the Inada conditions satisfied by CRRA utility. However, a short position in bonds ( $\alpha > 1$ ) is certainly plausible with a low CRRA parameter  $\gamma$  and a large equity premium  $\eta$  if the present value of “safe” future labor income (the deterministic component minus the upper support on the risky components) is large enough. This outcome has occurred in some previous papers but does not appear herein under our calibrations and modeling of social security.

<sup>11</sup>Thus, we take a somewhat conservative view on the value of carry forward provisions for loss offsets available in US tax law.

<sup>12</sup>Our state space, therefore, is about 100 times larger than most existing portfolio choice models due to our added complexities. Despite our choice of homothetic utility, our model is not homogenous of degree 1 due to the presence of nonlinear fiscal policies. We, therefore, cannot employ the standard technique of dropping the current state of income. We must also track the household’s average wage.

requirement is especially relevant for households without access to employer-based defined-contribution plans. Thus, in general, individual  $i$ 's maximization problem can be restated recursively as follows

$$V_{i,t}(X_{i,t}, z_{i,t}) = \max_{C_{i,t}, \alpha_{i,t}} \left\{ \frac{(C_{i,t})^{1-\gamma}}{1-\gamma} + \beta \left[ p_t \cdot E_t V_{i,t+1}(X_{i,t+1}, z_{i,t+1}) + b(1-p_t) \cdot E_t \left( \frac{(W_{i,t+1})^{1-\gamma}}{1-\gamma} \right) \right] \right\} \quad (14)$$

subject to<sup>13</sup>

$$\alpha_{i,t} \in [0, 1] \quad (15)$$

$$X_{i,t+1} = R_{i,t}(X_{i,t} - C_{i,t}) + (1 - h_{t+1})Y_{i,t+1}^d + SS_{i,t} \quad (16)$$

$$0 \leq \alpha_{i,t}(X_{i,t} - C_{i,t}) \notin (0, m) \quad (17)$$

where  $m \geq 0$  is set exogenously.

### 3.6 Numerical Solution

This maximization problem cannot be solved analytically. So numerical methods, using backward iteration over the time dimension, are used to derive the policy functions  $C_{i,t}(X_{i,t}, z_{i,t}, \bar{Y}_{i,t+1})$  and  $\alpha_{i,t}(X_{i,t}, z_{i,t}, \bar{Y}_{i,t+1})$ . The continuous state space is approximated using an unevenly spaced discrete grid (Appendix B). The value of the function  $V_{i,t+1}$  must be approximated between grid points across the three non-time dimensions; Schumaker shape-preserving quadratic splines are used inside the state exhibiting the most curvature (cash on hand) combined with bilinear interpolation in the other two dimensions.<sup>14</sup> There exist three labor income shocks ( $\omega_{i,t}$ ,  $\xi_t$  and  $\phi_{i,t}$ ) as well as uncertainty to the innovation of excess stock returns  $\eta_t$ . Gauss-Hermite quadrature is used to discretize these shocks into several nodes for the numerical integration that is required to compute expectations. Since the innovation to excess stock returns is assumed to be correlated with aggregate permanent income shock, however, a transformation to the Gauss-Hermite quadrature is required for the bivariate normal

<sup>13</sup>The additional constraint,  $C_{i,t} > 0$ , never binds in our model under the Inada conditions consistent with CRRA utility.

We also formally require that total savings at time  $t$  be positive (i.e.,  $(X_{i,t} - C_{i,t}) > 0$ ) to prevent any attempt to borrow against the "safe" portion of future labor income or social security benefits. Theoretically, this constraint should also never bind under the Inada conditions since, under our distributional assumptions, the individual  $i$ 's labor income in any year could be zero, as could the aggregate wage index factor. (In other words, there is actually no positive levels of "safe" labor or social security income.) In actual simulations, however, both of these terms must be bounded above zero when using standard numerical integration approximation methods. Still, the minimum integrated node is small enough to prevent negative savings.

<sup>14</sup>We thanks Ken Judd for this suggestion. Consistent with suggestions in Judd (1998), we found several other approximation methods to be very inaccurate, including Chebyshev polynomials. Schumaker splines does not generalize easily to more than one dimension.

distribution, as shown in Appendix B. Maximization at each grid point is performed using a combination of simplex optimization (the Nelder-Mead method) and Brent’s method (when feasible).<sup>15</sup>

Correctly imposing the minimum threshold constraint (17) requires some care. The optimization problem is first solved (using simplex optimization) at the given grid point without the constraint (17) imposed. If the constraint (17) is not violated then that solution is obviously used. Otherwise, two additional conditional optimization problems are solved (now using Brent’s method): where  $\alpha_{i,t}(X_{i,t} - C_{i,t}) = 0$  and where  $\alpha_{i,t}(X_{i,t} - C_{i,t}) = m$ . The best solution (highest value function) of those two optimizations is then chosen, which under our convexity assumptions, must be larger than any interior point where  $\alpha_{i,t}(X_{i,t} - C_{i,t}) > m$ . Solving individual  $i$ ’s lifetime maximization problem, therefore, requires up to 600,000,000 optimization subproblems to be solved, which requires about 250 CPU hours using a high-end processor and FORTRAN 90.<sup>16</sup> The simulation is solved with FORTRAN 90 MPI using a grid network where parallelization is conducted along the age (time) dimension.

However, to prevent us from simply picking parameter combinations that replicate the empirical stock allocation over the lifecycle, we require that any reported simulations also produce the correct wealth-to-income ratios over the lifecycle. This overidentifying restriction effectively creates an additional “outer loop” that requires testing many cases. Because each simulation, though, is very costly to compute, we manually “hunt and peck” at these parameters to generate wealth-income ratios that are “close enough” (as reported below).

Following Judd (1992, 1998), the accuracy of any simulation is likely best judged from the Euler equation errors produced by the approximation. Appendix C shows describes how we calculate the Euler errors for our model as a fraction of consumption at the closest grid point, thereby giving a meaningful (scale independent) interpretation to the errors. Except at corner constraints (where the Euler equation does not hold), the optimizations are required to satisfy a small sup norm error (Appendix C).

## 4 Calibration

Our model is calibrated to the United States economy in 2004, as described in this section. We distinguish individuals by their maximum amount of education obtained:  $i \in \{\text{“No High School”}, \text{“High School”}, \text{“College”}\}$ . Specifically, individuals with “No High School” do not have a high school diploma; individuals with “high school” have a high school diploma but not college; individuals with “College” have obtained a college degree.

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<sup>15</sup>Several other maximization methods and approaches were examined, including derivative-based methods as well as state-of-the-art quasi-Newton methods modified to ensure a correctly signed inverse Hessian matrix. Even a high-end commercial package was tested. However, none of them performed either as accurate (based on Euler Equation errors) or as fast as the approaches we eventually used.

<sup>16</sup>We reject any simulation (and, hence, try another approach) if *any* of the subproblems fail to solve. Hence, we are forced to use methods that are very stable and perform well globally.

## 4.1 Preferences

Individual  $i$  without a college degree starts working at age 20 while individuals with a college degree begin working at age 22. All households retire at age 65 and can live up to age 100. The mortality data for the whole population follow the National Vital Statistics Reports from National Center for Health Statistics (Elizabeth Arias 2004). The benchmark bequest intensity parameter  $b$  is set equal to 5 to help best calibrate the model.

## 4.2 Financial Assets

The riskfree return  $R_f$  is set equal to 2.00% while the average equity premium  $\mu$  is equal to 4.00% per year. The standard deviation of the innovation to excess returns  $\sigma_\eta$  is set to be the historical value 0.157.<sup>17</sup> When a minimum investment amount in stocks  $m$  is required to be positive, it is set equal to \$3000; this amount is fairly typical for no-load funds which are not connected to an employer-based retirement account. Empirically, the correlation of stocks and individual  $i$ 's wages at an annual frequency  $\rho_{y\eta}$  is fairly small; we will consider a value of 0.20 as an upper bound. The value of the correlation between the aggregate average wage and stock returns  $\rho_{\xi\eta}$ , however, is quite large at a low (30-year) frequency, although it is measured with considerable uncertainty. Specifically; a point estimate of 0.80 is consistent with Jermann (1999) and with our own estimates; however, the concomitant 95% confidence interval is quite large in light of the small number of data points. As a result, we consider a wide range of values for  $\rho_{\xi\eta}$ , from 0.15 to 0.80.

## 4.3 Labor Income and Housing Expenditures

Our labor income process follows the careful empirical analysis of Cocco, Gomes and Maenhout (CGM) (2005), who used the Panel Study of Income Dynamics (PSID) to estimate the deterministic function of labor income  $g(t, F_{i,t})$ , which they interpret broadly to include labor income, unemployment compensation, workers compensation, social security, supplemental social security, other welfare, child support, and total transfers from relatives.  $g(t, F_{i,t})$  is assumed to be additively separable in  $t$  and  $F_{i,t}$ , where age  $t$  is estimate as a third-order polynomial estimated to fit the age-wage profiles. They adopt the variances of transitory shocks and permanent shocks on labor income from Carroll (1997). The correlation coefficient between stock returns shocks and labor income shocks follows Campbell et al. (2001).<sup>18</sup>

The share of expenditures spent on housing  $h_t$  is taken from Gomes and Michaelides (2005), which they estimate using PSID data between 1976 and 1993

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<sup>17</sup>This value has been used in a couple previous papers; most of the literature uses a value between 0.15 - 0.20.

<sup>18</sup>Since the underlying data was in 1992 dollars while the fiscal policy part of our model is scaled to 2004 dollars, we use the change in the Consumer Price Index 34.11% as the inflation rate from 1992 to 2004.

as a function of age. Table 1 reports all the parameter values and coefficients. Table 2 illustrates the income process coefficients for different education groups.

#### 4.4 Progressive Federal Tax and Social Security Benefit Formula

We adopt the 2004 federal personal income tax rates (shown in Table 3 for single filers). Taxes on both risky and riskless asset returns  $\tau_d$  are set at proportional rate of 20%. The social security tax rate  $\tau_{SS}$  is 12.4%. After calculating the AIYE, we calculate the annual social security benefit using the following three-region “bend point” formula used by the U.S. Social Security system in 2004: 90 percent of the first \$7,344 of AIYE, plus 32 percent of the AIYE over \$7,344 through 44,268, plus 15 percent of the AIYE above 44,268.

### 5 Results for High School Education

After obtaining the optimal policy functions for each grid point on the state, we generate an initial distribution of random variables for labor income shocks and portfolio return shocks then simulate over 10,000 households. This section reports the average (mean) outcomes by age for those individuals in the middle education group: those people with a high school degree. We consider for four different calibrations for this education group: (i) a small value of aggregate average wage-stock correlation  $\rho_{\xi\eta}$  equal to 0.15 combined with no minimum constraint on investment returns  $m = \$0$ ; (ii) again,  $\rho_{\xi\eta} = 0.15$  but with a minimum constraint on investment returns  $m = \$3000$ ; (iii) a larger correlation  $\rho_{\xi\eta} = 0.80$  with no minimum investment  $m = \$0$ ; and (iv) a larger correlation  $\rho_{\xi\eta} = 0.80$  with a minimum investment  $m = \$3000$ . In each case, the value of risk aversion  $\gamma$  and discount rate  $\beta$  are chosen to create the closest possible match to the empirical wealth - labor income ratio.<sup>19</sup> The resulting parameter values are shown for in Table 4.

Figure 3 shows the average labor (plus social security) income and consumption profiles by age for the high school group both pre- and post-retirement for case (i). (The results for the other cases are qualitatively similar.) Average labor income takes the standard “hump shape” ( $\cap$ ) and drops to zero at retirement when Social Security benefits become positive. Consumption is also hump shaped, which is *not* standard in a *deterministic* lifecycle model where consumption should be monotonic in age. Intuitively, in the presence of income uncertainty and CRRA utility (where the third derivative in consumption is positive), individuals save for precautionary reasons in addition to standard retirement reasons. Hump-shaped consumption profiles emerge as a combination

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<sup>19</sup>To be sure, there is some subjectivity to this visual matching. However, at 250 CPU hours per simulation, we could not formally examine a maximum likelihood criterion along a large grid of parameter values. Still, we are fairly confident that we choose nearly the best combinations using a considerable range of trials. We decided to fix the bequest weight  $b = 5$  for all simulations to limit the search criterion and since this parameter is the least sensitive.

of precautionary savings and impatience, producing an optimal consumption path that tracks income (Zeldes 1989; Deaton 1991; Carroll 1992; Hubbard, Skinner and Zeldes 1994).

Figure 4 shows that the model’s simulated wealth-income ratios appears to match the data fairly closely for our four different calibrations. One obvious exception is oldest age group where survivor bias might play a role in the data. In particular, according to our simulations, wealthier households will hold a larger portion of the portfolio in equities. Empirically, these households tend to live longer than poorer households. Our simulations, however, do not correct for this wealth-longevity correlation.<sup>20</sup>

Figure 5 shows the share of savings allocated to stocks versus bonds for each of these cases. In each case, the stock allocation profile is fairly reasonable to the “hump shape” ( $\cap$ ) pattern in empirical data. Of course, the results are not a perfect fit by any means. As suggested earlier, however, the data has a fair amount of “noise” in it; we are more interested in replicating general portfolio “patterns” by age as closely as possible while generating realistic wealth-income restrictions. The hump shape is produced by the model in presence of several competing factors that vary by age. First, risky stocks and social security are highly substitutable at a young age where aggregate wage-indexed social security benefits are still quite uncertain. However, stocks and *bonds* become substitutable at a higher age as social security benefits accrue and the potential risk in the wage-indexed factor becomes. The net effect of this first factor is to cause the allocation in stocks to start relatively small at a young age and then increase in age. Second, similar to previous models, agent  $i$ ’s specific human capital depreciates over the lifecycle. Since stock returns and *individual* wages are *not* highly correlated, this effect produces a larger stock allocation at a young age that decreases with age – just the opposite pattern as the first factor. Third, the background risk caused by non-insurability of human capital returns interacting with the utility function’s Inada condition diminishes over the lifecycle, which can increase the share allocated toward risky stocks. Putting these competing factors together produces a “hump” shaped stock allocation.

However, that the plausibility of the needed the underlying parameters varies significantly between the simulations. Table 4 shows that the relatively smaller value of aggregate average wage-stock correlation  $\rho_{\xi\eta} = 0.15$  (Cases (i) and (ii)) requires  $\beta = 0.76$  and  $\gamma = 6$  to properly calibrate with the wealth-income ratio found in the data. In contrast, the relatively larger value of aggregate average wage-stock correlation  $\rho_{\xi\eta} = 0.80$  (Cases (iii) and (iv)) requires  $\beta = 0.86$  and  $\gamma = 4$  to properly calibrate. These differences imply substantially different tolerances toward risk as well as the discounting of future utility.

Notice that the presence of the minimum constraint  $m$  tends, if anything, to decrease the amount invested in stocks too much. This outcome is especially

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<sup>20</sup>While allowing for a wealth-longevity correlation would not require an additional state variable, it is actually fairly complicated when  $\gamma > 1$ , due to negative utility. In particular, *longer* longevity would actually *reduce* utility. In effect, allowing for longer longevity changes the preference parameters (the augmented utility weight) of an individual, making it difficult to compare.

important under our more plausible parameter cases when  $\rho_{\xi\eta} = 0.80$ . In Cases (iv), with the constraint, the portfolio share in equities at a young age is too low. In Case (iii), without the constraint, however, the portfolio share in equities early in life is very close to the reported empirical value. This result suggests that a minimum account balance is not very important, at least for the middle education group.

In the cases in which a larger weight is placed on future utility, a second “hump” later in the lifecycle is produced. The reason is fairly straightforward. After retirement – once the initial benefit amount is calculated – Social Security benefits are not adjusted for wage growth and only keep pace with inflation. Several of our simulations, therefore, suggest that the portfolio share devoted to stocks should potentially increase later in life, thereby creating an “echo” hump shape. This result appears to be counterfactual. From a normative perspective, however, they suggest just the opposite advice relative to conventional wisdom: the share of the portfolio devoted to equities should, if anything, *increase* as people approach retirement when social security benefits become safer. Indeed, the existing empirical evidence of portfolio holdings later in the lifecycle might potentially reflect this faulty conventional wisdom.

In fact, conventional financial planning – which argues that stocks holdings should decrease almost linearly over the lifecycle – might be completely misguided on both sides of the age spectrum. Younger households should potentially hold fewer equities, consistent with the empirical evidence, whereas many older households should hold more equities, which is not consistent with the empirical evidence. One possible reconciliation between theory and practice is that older households, with more assets at risk, are more sensitive to conventional financial planning advice than younger households, who have fewer assets at risk. This conjecture strikes us as fairly plausible and is consistent with anecdotal evidence.

## 6 No High School and College

The results for households with college education (Figures 6 and 7) are similar except that they tend to hold more equities, at the same age, relative to the middle group considered above who only have a High School diploma. Similarly, on the opposite extreme, households Without High School (Figures 8 and 9) tend to hold fewer equities, at the same age, relative to the High School group.

These results are generated by the progressive nature of the U.S. Social Security system: the Social Security replacement ratio for poorer households is substantially larger than for richer households. Although future income is stochastic, households still make rational expectations based on the moments of the underlying processes. As a result, Social Security benefits tend to crowd out the savings for younger households that are poorer, which is a conventional result. However, for poor households, Social Security will also crowd out more of the amount of their savings that is devoted to stocks due to the correlation with Social Security at low frequency.

## 7 Conclusion

This paper examines how households should optimally allocate their portfolio choices between stocks and bonds in a lifecycle model where they face uninsurable wage shocks and an uncertain lifetime. We include a progressive social security system that is *wage indexed* before retirement, as in many pay-as-you-go systems including the United States. Social security benefits are correlated with stock returns at a low frequency that is relevant for lifecycle retirement planning. We show that this model is able to more closely replicate the key stylized facts of portfolio choice relative to alternative specifications.

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## Appendix A

To construct the the household income, we include wages and salaries (X5702), practice/business/partnership/farm income (X5704), rent/trusts/royalties (X5714), unemployment or worker's compensation (X5716), child support or alimony (X5718), food stamps and welfare income (X5720) and other income (X5724).

Bonds and stocks are constructed as follows. All the acronym-variables are defined in the SAS program supplied by the SCF, which creates summary variables for SCF. Bonds consist of SAVING and MMA (savings and money market accounts), CDS (certificates of deposit), TFBMUTF (tax free bond mutual funds), GBMUTF (government bond mutual funds), OBMUTF (other bond mutual funds), BOND (state, US government and corporate bonds), SAVBND (saving bonds) and COMUTF (combination and other mutual funds), for which we assume that half is invested to bonds. We also add ANNUIT (annuities) and TRUSTS (trusts), for which we count the full value if the individuals invest all in interest earning assets, while the percentage other than stock allocation if the individuals split the investment. Other bond investment includes bonds in IRA/KEOGH plans, bonds in account-type retirement plans and FUTPEN (other future pension benefits). We also subtract CCBAL (revolving credit card debt), OTHLOC (unsecured loans and loans secured by pensions) and other debt, which includes loans against pensions (X11027, X11127, X11327, X11427, X11527), loans against life insurance (X4010) and loans against margin loans (X3932).

Stocks are made up of STOCKS (directly held stocks), STMUTF (stock mutual funds), half of COMUTF (combination and other mutual funds), OMUTF (other non-bond mutual funds), PENEQ (thrift amounts invested in stock), ANNUIT (annuities) and TRUSTS (trusts) that are invested in stocks. Other stock investment includes stocks in IRA/KEOGH plans, stocks in account-type retirement plans and FUTPEN (other future pension benefits).

The total financial wealth is defined as the total of bonds, stocks, CHECKING (checking accounts) and CALL (call accounts).

## Appendix B

$f(\cdot)$  denotes the p.d.f. of the variables.  $I, J, K, L$  are the numbers of interpolation nodes for labor income shocks and stock returns shocks  $\eta, \xi, \omega, \phi$ , respectively. The distributions of the shocks are summarized as,

$$\omega \xrightarrow{d} N(\mu_\omega, \sigma_\omega^2) \quad (18)$$

$$\phi \xrightarrow{d} N(\mu_\phi, \sigma_\phi^2) \quad (19)$$

$$\begin{pmatrix} \eta \\ \xi \end{pmatrix} \xrightarrow{d} N \begin{pmatrix} \mu_\eta & \sigma_\eta^2 & \rho_{\xi\eta}\sigma_\xi\sigma_\eta \\ \mu_\xi & \rho_{\xi\eta}\sigma_\xi\sigma_\eta & \sigma_\xi^2 \end{pmatrix} \quad (20)$$

For each individual  $i$ , at any time  $t$  and any state point  $(X_{i,t}, z_{i,t})$ , the expectation of the value function can be expressed as follows.

$$\begin{aligned} E_{\xi\eta\omega\phi} V_t(\xi, \eta, \omega, \phi, X_t, z_t) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} V_t(\xi, \eta, \omega, \phi, X_t, z_t) f(\xi, \eta, \omega, \phi) d\xi d\eta d\omega d\phi \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} V_t(\xi, \eta, \omega, \phi, X_t, z_t) f(\eta|\xi) d\eta \cdot f(\xi) d\xi \cdot f(\omega) d\omega \cdot f(\phi) d\phi \end{aligned} \quad (21)$$

For the bivariate normal distribution, the conditional distribution for one of the variables, given the value for the other variable, is normally distributed. Therefore,

$$\eta|\xi \xrightarrow{d} N(\mu_{\eta|\xi}, \sigma_{\eta|\xi}^2) \quad (22)$$

where

$$\mu_{\eta|\xi} = \mu_\eta + \frac{\rho_{\xi\eta}\sigma_\eta}{\sigma_\xi} (\xi - \mu_\xi) \quad (23)$$

$$\sigma_{\eta|\xi}^2 = \sigma_\eta^2 (1 - \rho_{\xi\eta}^2) \quad (24)$$

Following P.262 of Kenneth Judd, if a normal random variable  $X$  is distributed  $N(\mu, \sigma^2)$ , then the general Gauss-Hermite quadrature rule for expectation of  $X$  is as follows,

$$E(X) = \pi^{-1/2} \sum_{i=1}^n w_i \cdot (\sqrt{2}\sigma x_i + \mu)$$

where  $w_i$  are the Gauss-Hermite quadrature weights, and  $x_i$  are the quadrature nodes,  $i = 1, 2, \dots, n$ .

So, the functional approximation of Gauss-Hermite quadrature to  $\eta$  is

$$\begin{aligned} E_{\xi\eta\omega\phi} V_t(\xi, \eta, \omega, \phi, X_t, z_t) &= \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{i=1}^I w_{i,\eta} V_t(\xi, \sqrt{2}\sigma_\eta \xi + \mu_{\eta|\xi}, \omega, \phi, X_t, z_t) \\ &\quad \cdot f(\xi) d\xi \cdot f(\omega) d\omega \cdot f(\phi) d\phi \\ &= \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{i=1}^I w_{i,\eta} V_t(\xi, \sqrt{2}\sigma_\eta \sqrt{1 - \rho_{\xi\eta}^2} \eta_i + \mu_\eta + \\ &\quad \frac{\rho_{\xi\eta} \sigma_\eta}{\sigma_\xi} (\xi - \mu_\xi), \omega, \phi, X_t, z_t) \cdot f(\xi) d\xi \cdot f(\omega) d\omega \cdot f(\phi) d\phi \end{aligned} \quad (25)$$

where  $w_{i,\eta}$  is the weights,  $\eta_i$  is the nodes over  $[-\infty, \infty]$

The procedure of discretizing  $\xi$  is to substitute  $\sqrt{2}\sigma_\xi \xi_j + \mu_\xi$  for  $\xi$ , which yields the following,

$$\begin{aligned} &E_{\xi\eta\omega\phi} V_t(\xi, \eta, \omega, \phi, X_t, z_t) \\ &= \pi^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{j=1}^J \sum_{i=1}^I w_{i,\eta} w_{j,\xi} V_t(\sqrt{2}\sigma_\eta (\sqrt{1 - \rho_{\xi\eta}^2} \eta_j + \rho_{\xi\eta} \xi_j) + \mu_\eta, \omega, \phi, X_t, z_t) \cdot f(\omega) d\omega \cdot f(\phi) d\phi \end{aligned} \quad (26)$$

Again  $w_{j,\xi}$  and  $\xi_j$  are the Gauss-Hermite quadrature weights and nodes, respectively. Proceeding with the integral over  $\omega$  and  $\phi$ ,

$$\begin{aligned} &E_{\xi\eta\omega\phi} V_t(\xi, \eta, \omega, \phi, X_t, z_t) \\ &= \pi^{-2} \sum_{l=1}^L \sum_{k=1}^K \sum_{j=1}^J \sum_{i=1}^I w_{i,\eta} w_{j,\xi} w_{k,\omega} w_{l,\phi} \cdot V_t(\sqrt{2}\sigma_\eta (\sqrt{1 - \rho_{\xi\eta}^2} \eta_j + \rho_{\xi\eta} \xi_j) + \mu_\eta, \sqrt{2}\sigma_\omega \omega_k \\ &\quad + u_\omega, \sqrt{2}\sigma_\phi \phi_l + u_\phi; X_t, z_t) \end{aligned} \quad (27)$$

This model is expensive to compute. The number of nodes in the dynamic programming tree is  $I \times J \times K \times L \times T \times S_1 \times S_2 \times S_3 \times 2$ , where  $I, J, K, L$  are the nodes for different shocks as described above,  $T$  is the maximum age that individual can live up to,  $S_1, S_2$  and  $S_3$  are the numbers of nodes for three

state variables separately. Taking 80 as the total life periods and 5 nodes for each shock, the number of nodes for cash on hand is 60, while the number of nodes for income and indexed wage is 10, separately. We have two situations for each optimization problem. One is with bond investment only, the other is with both riskless and risky allocation. Therefore,  $I \times J \times K \times L \times T \times S_1^{\max} \times S_2^{\max} \times S_3^{\max} \times 2 = 600,000,000$ .

The grid nodes are discretized as follows. Let  $N_X$  represent the grid vector for cash on hand. The minimum {TO DO}

### Appendix C

Take the first-order condition of Bellman equation

$$V_{i,t}(X_{i,t}, z_{i,t}) = \max_{C_{i,t}, \alpha_{i,t}} \{U(C_{i,t}) + \beta [p_t \cdot E_t V_{i,t+1}(X_{i,t+1}, z_{i,t+1}) + b(1-p_t) \cdot E_t U(W_{i,t+1})]\} \quad (28)$$

Subject to

$$X_{i,t+1} = R_{i,t}(X_{i,t} - C_{i,t}) + (1 - h_{t+1})Y_{i,t+1}^d \quad (29)$$

$$X_{i,t} = W_{i,t} + (1 - h_t)Y_{i,t}^d \quad (30)$$

We have

$$\begin{aligned} & \frac{\partial V_{i,t}(X_{i,t}, z_{i,t})}{\partial C_{i,t}} \quad (31) \\ = & U'(C_{i,t}) + \beta \{p_t \cdot E_t [V_{i,t+1}'(X_{i,t+1}, z_{i,t+1}) \cdot (-R_{i,t})] + b(1-p_t) \cdot E_t [U'(W_{i,t+1}) \cdot (-R_{i,t})]\} \\ = & 0 \quad (33) \end{aligned}$$

Change the equation, we have

$$U'(C_{i,t}) = \beta \{p_t \cdot E_t [V_{i,t+1}'(X_{i,t+1}, z_{i,t+1}) \cdot R_{i,t}] + b(1-p_t) \cdot E_t [U'(W_{i,t+1}) \cdot R_{i,t}]\} \quad (34)$$

According to the Envelope Theorem, the partial derivative with respect to  $X_{i,t}$  is

$$\frac{\partial V_{i,t}(X_{i,t}, z_{i,t})}{\partial X_{i,t}} = \beta \{p_t \cdot E_t [V_{i,t+1}'(X_{i,t+1}, z_{i,t+1}) \cdot R_{i,t}] + b(1-p_t) \cdot E_t [U'(W_{i,t+1}) \cdot R_{i,t}]\} \quad (35)$$

The right hand sides of Equations (34) and (35) are equal, which gives us

$$U'(C_{i,t}) = V_{i,t}'(X_{i,t}, z_{i,t}) \quad (36)$$

Thus, we rewrite Equation (34) as follows,

$$U'(C_{i,t}) = \beta p_t \cdot E_t [U'(C_{i,t+1}) \cdot R_{i,t}] + \beta b(1-p_t) \cdot E_t [U'(W_{i,t+1}) \cdot R_{i,t}] \quad (37)$$

$$C_{i,t} = U'^{-1} \{ \beta p_t \cdot E_t [U'(C_{i,t+1}) \cdot R_{i,t}] + \beta b(1-p_t) \cdot E_t [U'(W_{i,t+1}) \cdot R_{i,t}] \} \quad (38)$$

We define the Euler error as

$$C_{i,t} \cdot (1 + \text{error}) = U^{-1} \{ \beta p_t \cdot E_t [U'(C_{i,t+1}) \cdot R_{i,t}] + \beta b(1 - p_t) \cdot E_t [U'(W_{i,t+1}) \cdot R_{i,t}] \} \quad (39)$$

$$\text{error} = \frac{U^{-1} \{ \beta p_t \cdot E_t [U'(C_{i,t+1}) \cdot R_{i,t}] + \beta b(1 - p_t) \cdot E_t [U'(W_{i,t+1}) \cdot R_{i,t}] \} - C_{i,t}}{C_{i,t}} \quad (40)$$

We then take a  $\log_{10}$  of the *error*, the acceptable range is around -3 or smaller.

**Table 1: Common Parameters in All Simulations**

Description	Parameter Value
Start working age ( $t_0$ )	20/22
Retirement age ( $M$ )	65
Discount factor ( $\beta$ )	see text
Coefficient of relative risk aversion ( $\gamma$ )	see text
Riskless returns ( $R_f$ )	1.02
Mean risky returns ( $R_f + \mu$ )	1.06
Variance of transitory income shocks ( $\sigma_\omega^2$ )	0.01
Variance of permanent income shocks ( $\sigma_u^2$ )	0.01
Standard deviation of stock returns ( $\sigma_\eta$ )	0.157
Correlation between stock returns and income shocks ( $\rho_{\xi\eta}$ )	0.15
Social security tax rate ( $\tau_{SS}$ )	0.124
Investment return tax rate ( $\tau_{SS}$ )	0.20
Bequest intensity ( $b$ )	5
Minimum investment required in stock account ( $m$ )	\$3000 (when on)
Marital status	single
Family size	1

**Table 2: Labor Income Process**

Coefficient of characteristic variables for labor income	No high school	High school	College
Constant	2.6275	2.7004	2.3831
Marital status	0.4008	0.4437	0.4831
Family size	-0.0176	-0.0236	-0.0228
Coefficient of age dummies for labor income			
Constant	-2.1361	-2.1700	-4.3148
Age	0.1684	0.1682	0.3194
Age <sup>2</sup> /10	-0.0353	-0.0323	-0.0577
Age <sup>3</sup> /100	0.0023	0.0020	0.0033

**Table 3: 2004 Progressive Federal Personal Income Tax Rates**

Tax Rate	Single Filers
10%	Up to \$7,150
15%	\$7,151 - \$29,050
25%	\$29,051 - \$70,350
28%	\$70,351 - \$146,750
33%	\$146,751 - \$319,100
35%	\$319,101 or more

**Table 4: Calibrated Parameters for High School Group in 4 Cases**

	No Minimum Investment $m = \$0$	Minimum Investment $m = \$3000$
$\rho_{\xi\eta} = 0.15$	(i): $\beta = 0.76, \gamma = 6$	(ii): $\beta = 0.76, \gamma = 6$
$\rho_{\xi\eta} = 0.80$	(iii): $\beta = 0.86, \gamma = 4$	(iv): $\beta = 0.86, \gamma = 4$

**Table 5: Calibrated Parameters for College Group in 4 Cases**

	No Minimum Investment $m = \$0$	Minimum Investment $m = \$3000$
$\rho_{\xi\eta} = 0.15$	(i): $\beta = 0.60, \gamma = 6$	(ii): $\beta = 0.60, \gamma = 6$
$\rho_{\xi\eta} = 0.80$	(iii): $\beta = 0.74, \gamma = 4$	(iv): $\beta = 0.74, \gamma = 4$
<b>Table 6: Calibrated Parameters for No-High School Group in 4 Cases</b>		
	No Minimum Investment $m = \$0$	Minimum Investment $m = \$3000$
$\rho_{\xi\eta} = 0.15$	(i): $\beta = 0.86, \gamma = 6$	(ii): $\beta = 0.86, \gamma = 6$
$\rho_{\xi\eta} = 0.80$	(iii): $\beta = 0.94, \gamma = 4$	(iv): $\beta = 0.94, \gamma = 4$

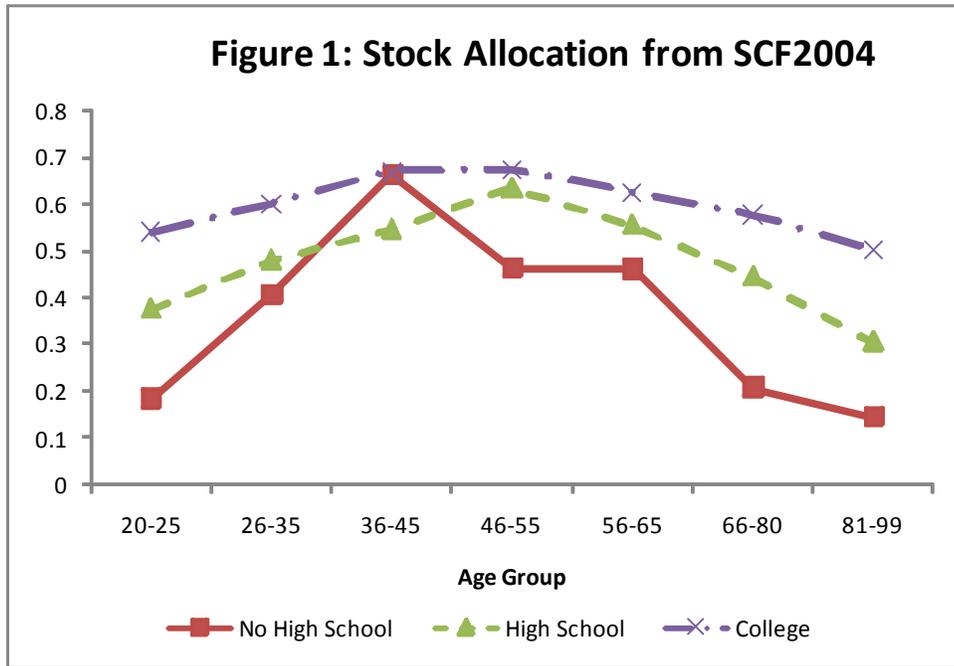


Figure 1:

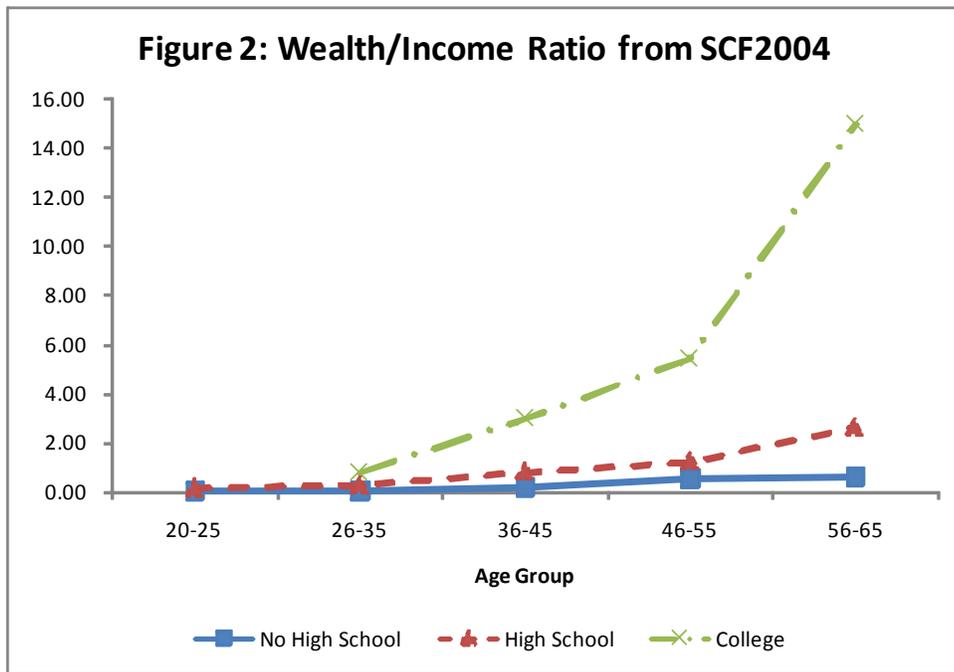


Figure 2:

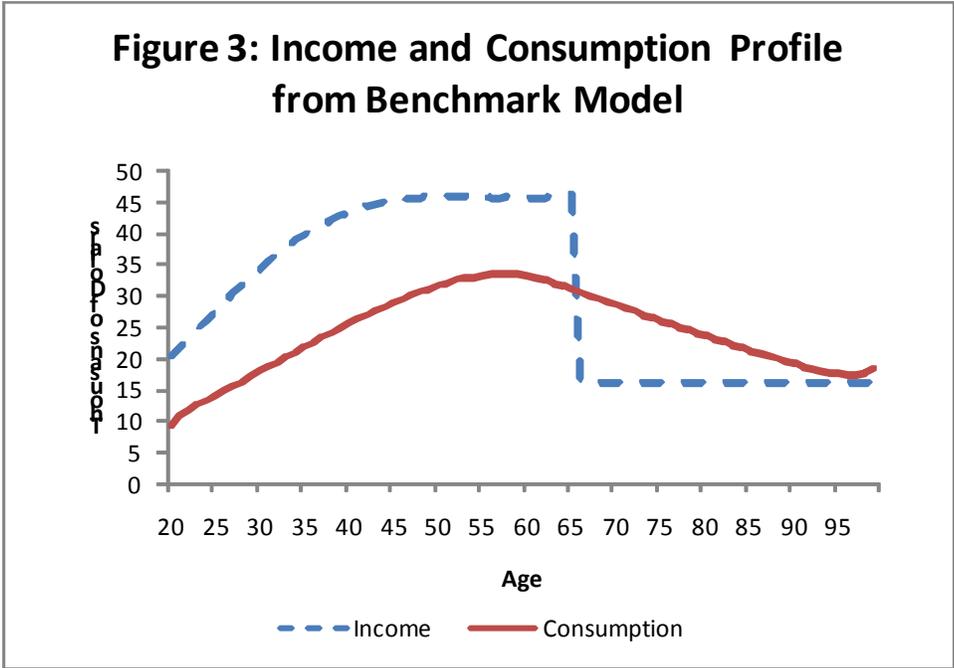


Figure 3:

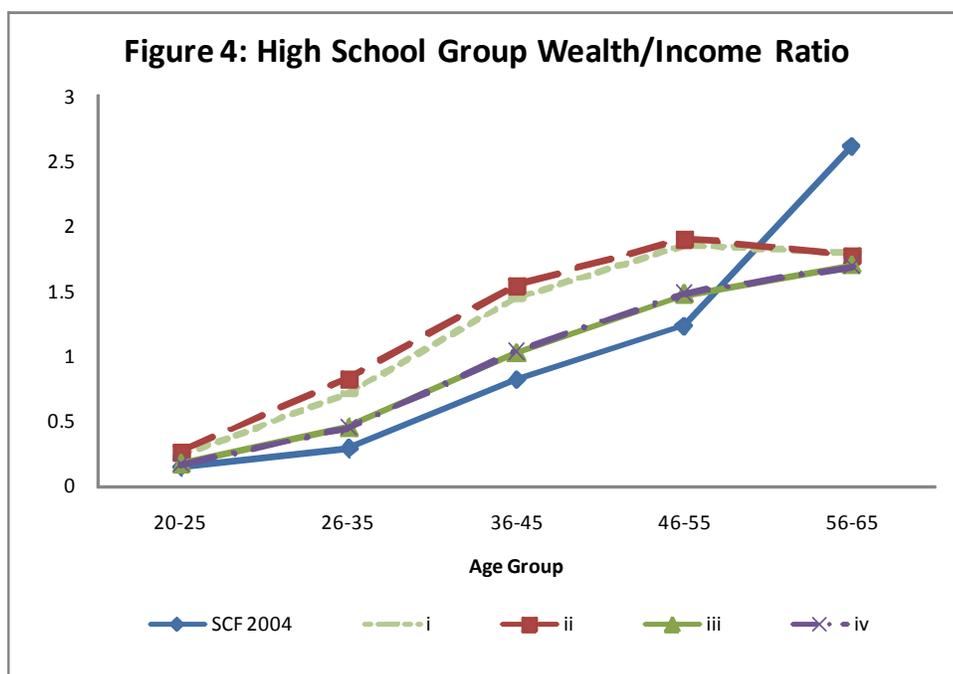


Figure 4:

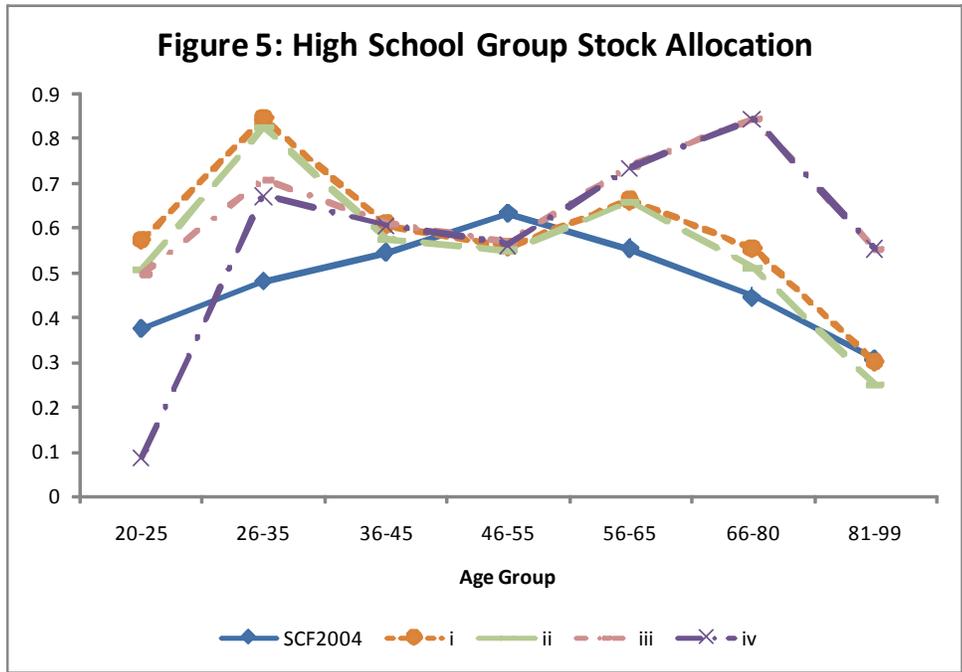


Figure 5:

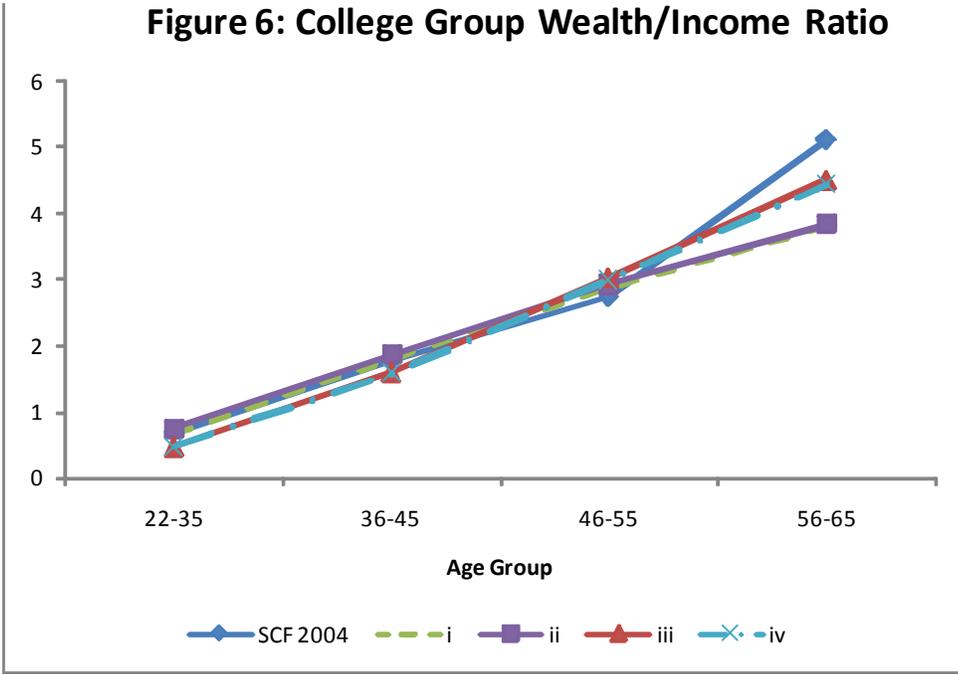


Figure 6:

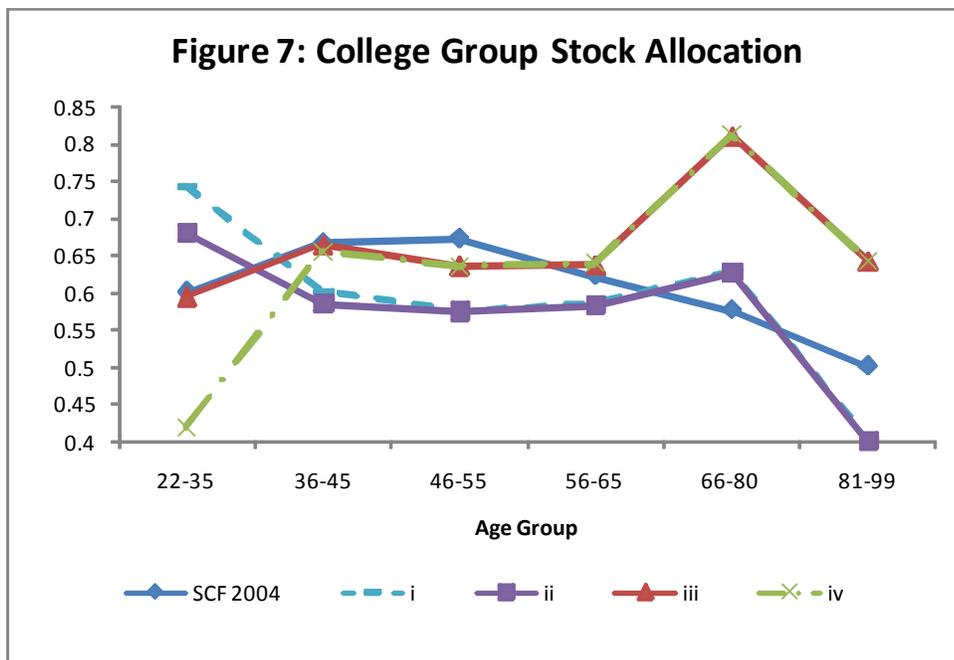


Figure 7:

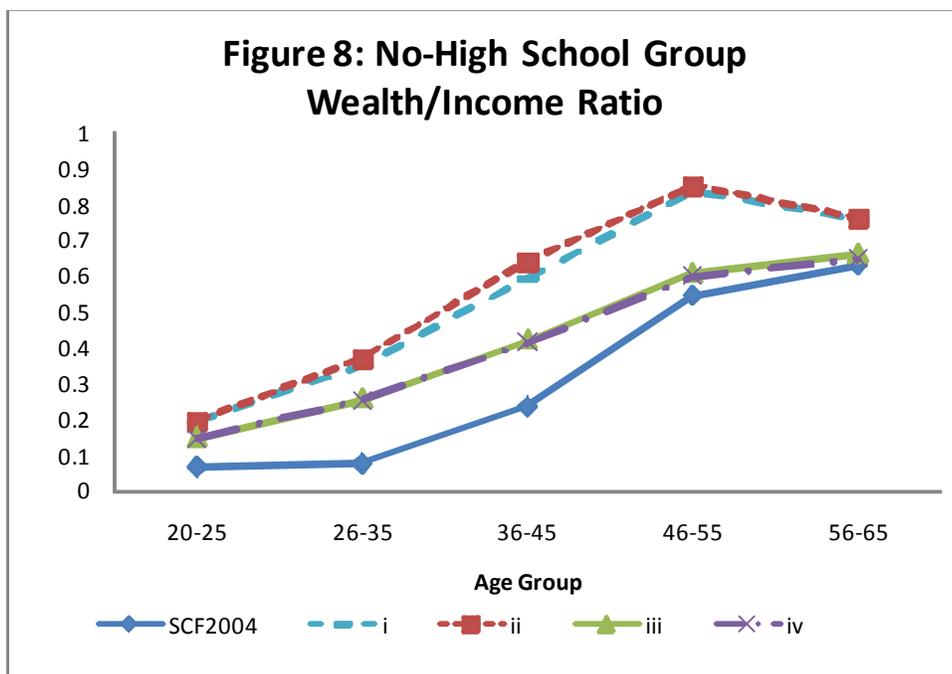


Figure 8:

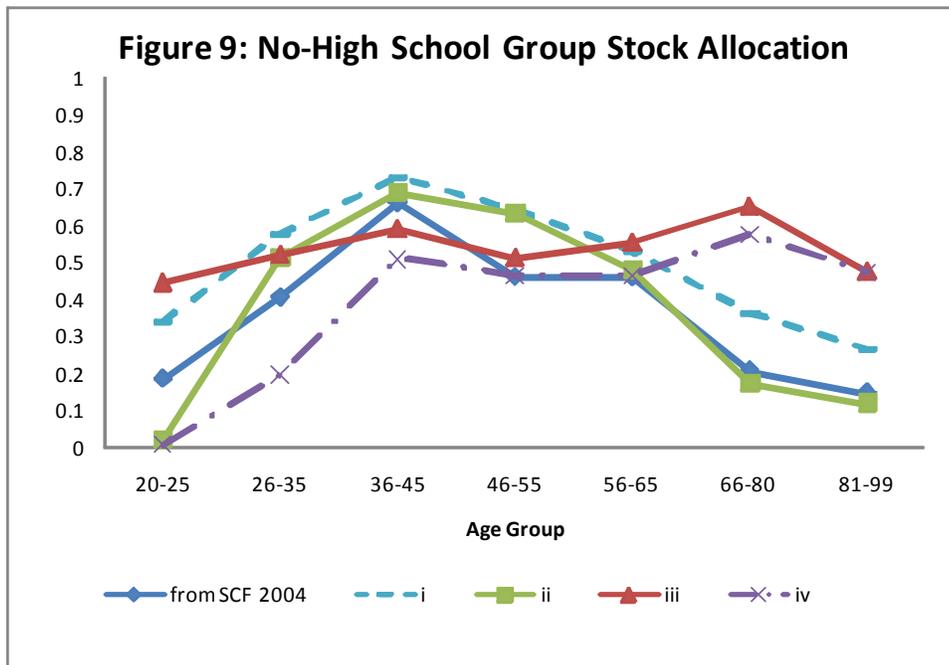


Figure 9: