

Risk Premia and the Conditional Tails of Stock Returns

Bryan Kelly

NYU Stern and Chicago Booth

Outline

- **Introduction**
- An Economic Framework
- Econometric Methodology
- Empirical Findings
- Conclusions

Tail Risk in the Big Picture

- Value of assets depends on the potential for infrequent, extreme payoff events
 - ① Peso problems (Krasker, 1980)
 - ② Potential for rare disasters can explain equity puzzles (Rietz, 1988; Barro, 2006; Weitzman 2007; Gabaix 2009; Wachter 2009)
- Plausible mechanism – OR – convenient (though unrealistic) explanation?

The Trouble with Tail Risk

- Tail risk is difficult to measure, even unconditionally
- Few risks are static: Feasibility of conditional tail measures?
- **My solution:** An economically-motivated **conditional tail risk measure** extracted from the *cross section* of asset returns

Objectives

- 1 Structural understanding of how tail risk is priced
 - I derive tractable expressions for expected returns as a function of a **tail risk state variable**
 - I derive the distribution of return tail events implied by the model
- 2 I econometrically identify the conditional tail distribution of returns
 - Directly estimable from the cross section of asset prices by exploiting restriction implied by economic theory
- 3 I evaluate theories relating **tail risk** to **risk premia** using my estimated series

Preview of Empirical Results

- Tail risk varies substantially over time and is highly persistent
- Tail measure predicts market returns over horizons of one month to five years, outperforms commonly studied predictors
 - A one standard deviation increase in tail risk increases expected returns by 4.4% per year
- Large explanatory power for cross section of returns
 - Stocks that covary highly with tail risk earn annual expected returns 2% to 6% lower than stocks that with low tail risk covariation

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Structural Models and Tail Risk

- Emergence in varied theoretical settings, for example
 - ① Long run risks + heavy-tailed shocks (similar to Eraker and Shaliastovich 2008, Drechsler and Yaron 2009)
 - ② Time-varying rare disasters (similar to Gabaix 2009, Wachter 2009)
 - ③ Long run risks + large swings in confidence (similar to Bansal and Shaliastovich 2009)

A Tail Risk State Variable in the Long Run Risks Framework

- Epstein-Zin preferences:

$$m_{t+1} = \theta \ln \beta - \frac{\theta}{\psi} \Delta c_{t+1} + (\theta - 1) r_{c,t+1}$$

- Dynamics of the real economy:

$$\Delta c_{t+1} = \mu + x_t + \sigma_c \sigma_t z_{c,t+1} + \sqrt{\Lambda_t} W_{c,t+1}$$

$$x_{t+1} = \rho_x x_t + \sigma_x \sigma_t z_{x,t+1}$$

$$\sigma_{t+1}^2 = \bar{\sigma}^2 (1 - \rho_\sigma) + \rho_\sigma \sigma_t^2 + \sigma_\sigma z_{\sigma,t+1}$$

$$\Lambda_{t+1} = \bar{\Lambda} (1 - \rho_\Lambda) + \rho_\Lambda \Lambda_t + \sigma_\Lambda z_{\Lambda,t+1}$$

$$\Delta d_{i,t+1} = \mu_i + \phi_i \Delta c_{t+1} + \sigma_i \sigma_t z_{i,t+1} + q_i \sqrt{\Lambda_t} W_{i,t+1}$$

$$f_W(w) = \frac{1}{2} \exp(-|w|), w \in \mathbb{R}$$

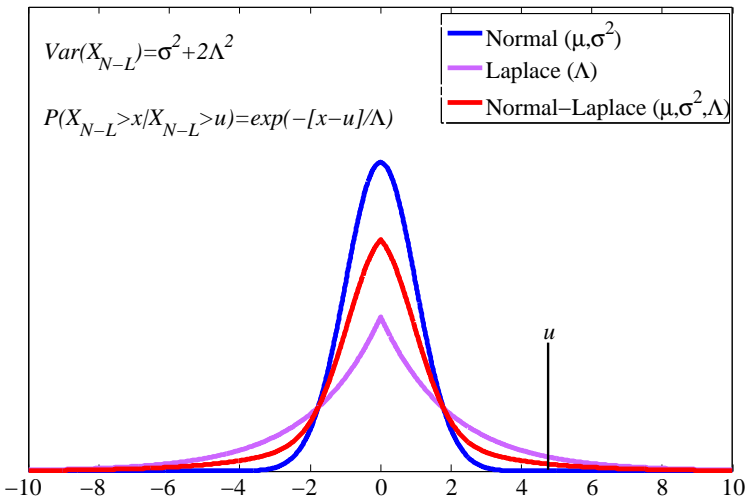
What Is Tail Risk?

- Gaussian baseline: is variance sufficient to characterize risk of extreme events?
- Illustrative example: Normal-Laplace distribution
- Since at least Mandelbrot (1963) and Fama (1963), economists have argued for power law return tails

$$P(R > x | R > u) = \left(\frac{x}{u}\right)^{-\zeta}, \quad u \text{ some high threshold}$$

- $-\zeta \equiv$ tail risk measure
- Does ζ change through time?
- What kind of world?

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$$f_W(w) = \frac{1}{2} \exp(-|w|), w \in \mathbb{R}$$

Prices and Excess Returns

Proposition

The log wealth-consumption ratio and log price-dividend ratio for asset (i) are linear in state variables,

$$wc_{t+1} = A_0 + A_x x_{t+1} + A_\sigma \sigma_{t+1}^2 + A_\Lambda \Lambda_{t+1}$$

$$pd_{i,t+1} = A_{i,0} + A_{i,x} x_{t+1} + A_{i,\sigma} \sigma_{t+1}^2 + A_{i,\Lambda} \Lambda_{t+1}.$$

Proposition

The expected return on asset (i) in excess of the risk free rate is

$$E_t[r_{i,t+1} - r_{f,t}] = \beta_{i,c} \lambda_c (\sigma_c^2 \sigma_t^2 + 2\Lambda_t) + \beta_{i,x} \lambda_x \sigma_x^2 \sigma_t^2 + \beta_{i,\sigma} \lambda_\sigma \sigma_\sigma^2 + \beta_{i,\Lambda} \lambda_\Lambda \sigma_\Lambda^2 - \frac{1}{2} \text{Var}(r_{i,t+1}).$$

► Proof

Key Implications

- ① Tail risk forecasts excess stock returns
High tail risk \Rightarrow high future returns
- ② Covariance with tail risk impacts cross section of expected returns
High return tail risk beta \Rightarrow low expected returns

Implied Distribution of Returns

Proposition

The lower and upper tail distributions of arithmetic returns are asymptotically equivalent to a power law,

$$P_t(R_{i,t+1} < r \mid R_{i,t+1} < u) \sim \left(\frac{r}{u}\right)^{a_i \zeta_t}$$

$$P_t(R_{i,t+1} > r \mid R_{i,t+1} > u) \sim \left(\frac{r}{u}\right)^{-a_i \zeta_t}$$

where $a_i = \max(\phi_i, q_i)^{-1}$ and $\zeta_t = 1/\sqrt{\Lambda_t}$.

Key Implications

Tail risk state variable drives **risk premia** and **tail exponent**

- 1 Tail risk (and thus tail exponent) forecasts excess stock returns
High tail risk \Rightarrow high future returns
- 2 Covariance with tail risk (and thus tail exponent) impacts cross section of expected returns
High return tail risk beta \Rightarrow low expected returns

▶ Other Structural Models

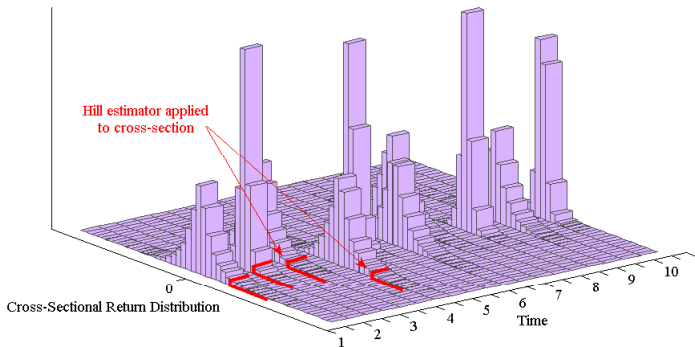
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Econometric Intuition

$$P_t(R_{i,t+1} < r \mid R_{i,t+1} < u) \sim \left(\frac{r}{u}\right)^{a_i \zeta_t}$$

- Single process drives tail dynamics of entire **panel** of returns



Definition: Dynamic Power Law Model

Individual returns on asset (i), conditional upon exceeding threshold u and given \mathcal{F}_t , obey

$$F_{u,i,t}(r) = P(R_{i,t+1} > r | R_{i,t+1} > u, \mathcal{F}_t) = \left(\frac{r}{u}\right)^{-a_i \zeta_t}$$

with exponent

$$\frac{1}{\zeta_{t+1}} = \pi_0 + \pi_1 \frac{1}{\zeta_t^{upd}} + \pi_2 \frac{1}{\zeta_t}$$

and observable update

$$\frac{1}{\zeta_t^{upd}} = \frac{1}{K_t} \sum_{k=1}^{K_t} \ln \frac{R_{k,t}}{u}$$

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with exponent

$$\frac{1}{\zeta_{t+1}} = \pi_0 + \pi_1 \frac{1}{\zeta_t^{upd}} + \pi_2 \frac{1}{\zeta_t} = \frac{\pi_0}{1 - \pi_2} + \pi_1 \sum_{j=0}^{\infty} \pi_2^j \frac{1}{\zeta_{t-j}^{upd}}$$

and observable update

$$\frac{1}{\zeta_t^{upd}} = \frac{1}{K_t} \sum_{k=1}^{K_t} \ln \frac{R_{k,t}}{u} \left\{ \begin{array}{l} \text{Hill (1975) Estimator} \\ \text{applied to cross section} \end{array} \right.$$

Quasi-Likelihood Estimator for Dynamic Power Law Model

Proposal: Assume tail observations in time t cross section are identical and independent

- But theory (and years of empirical work) suggests...
 - 1 Dependent observations (factor structure)
 - 2 Heterogeneous volatility
 - 3 Heterogeneous tail exponent

Result: Despite mis-specification, estimator *consistent* and *asymptotically normal*

Quasi-Likelihood Estimator for Dynamic Power Law Model

- 1 Assume (provisionally) tail observations are cross-sectionally independent and each obey

$$\tilde{F}_{u,i,t}(X_{i,t+1}; \pi) = \left(\frac{x}{u}\right)^{-\tilde{\zeta}_t}$$

$$\tilde{f}_{u,i,t}(X_{i,t+1}; \pi) = \frac{\zeta_t}{u} \left(\frac{x}{u}\right)^{-(1+\zeta_t)}$$

- 2 Construct log quasi-likelihood using only u -exceedences

$$\mathcal{L}(X; \pi) = \frac{1}{T} \sum_{t=0}^T \ln \tilde{f}_{u,t}(X_{t+1}; \pi) = \frac{1}{T} \sum_{t=0}^T \sum_{k=1}^{K_{t+1}} \left(\frac{1}{\tilde{\zeta}_t} - \ln \frac{X_{k,t+1}}{u} \right)$$

- 3 Maximize

$$\text{QML Estimator: } \hat{\pi}_{QL} \equiv \arg \max_{\pi \in \Pi} \mathcal{L}(X; \pi)$$

Asymptotic Properties of QML Estimator

Proposition

Let the true DGP of $\{R_t\}_{t=1}^T$ be given by the Dynamic Power Law model with parameter values π^* . Under standard GMM regularity conditions,

$$\hat{\pi}_{QL} \xrightarrow{P} \pi^*$$

and

$$\sqrt{T}(\hat{\pi}_{QL} - \pi^*) \xrightarrow{d} N(0, \Psi)$$

where

$$\Psi = S^{-1}GS^{-1}, S = E[\nabla_{\pi}s(X_t; \pi^*)], \text{ and } G = E[s(X_t; \pi^*)s(X_t; \pi^*)'].$$

Proof Sketch

- First order condition of quasi-likelihood maximization

$$s(X_{t+1}; \pi) \equiv \nabla_{\pi} \ln \tilde{f}_t(X_{t+1}; \pi) = \frac{K_{t+1}}{\tilde{\zeta}_t} - \sum_{k=1}^{K_{t+1}} \ln \frac{X_{k,t+1}}{u} = 0$$

- MLE identification condition: expected value of FOC equals zero
- Mis-specified MLE is GMM – FOC moment condition holds

Lemma

$E[s(X_{t+1}; \pi)] = 0$ when the true model is the Dynamic Power Law.

$$\begin{aligned} \text{Proof: } E[s(X_{t+1}; \pi)] &= E\left[E_t\left[\frac{K_{t+1}}{\tilde{\zeta}_t} - \sum_{k=1}^{K_{t+1}} \ln \frac{X_{k,t+1}}{u}\right]\right] \\ &= E\left[\frac{K_{t+1}}{\tilde{\zeta}_t} - \frac{K_{t+1} \frac{1}{n} \sum_i a_i}{\zeta_t}\right] \\ &= 0 \text{ when } \frac{1}{\tilde{\zeta}_t} = \frac{1}{n} \sum_i a_i. \quad \square \end{aligned}$$

Volatility and Other Considerations

- By varying threshold each period, accommodate time-varying volatility
- Cross sectional differences in volatility?
- Explicitly modeling dependence?

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Data

- Primary sample: Daily NYSE/AMEX/NASDAQ stock returns from CRSP
- Fama-French factors; Ken French's data library
- Federal Reserve macro data
- Goyal and Welch (2008) data
- OptionMetrics
- Other (VIX, Hao Zhou's variance risk premium)

▶ Count

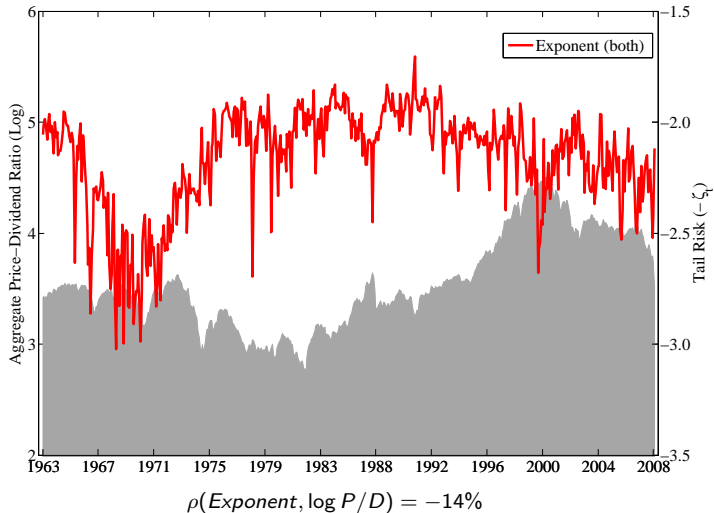
Dynamic Power Law Estimates

$$\frac{1}{\zeta_{t+1}} = \pi_0 + \pi_1 \frac{1}{\zeta_t^{upd}} + \pi_2 \frac{1}{\zeta_t}$$

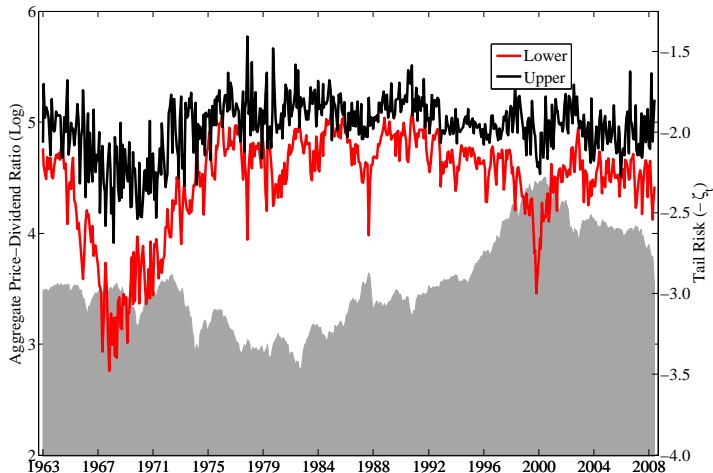
Table: 1963-2008

	Both Tails	Lower Tail	Upper Tail
$\bar{\zeta}$	2.110 (0.021)	2.201 (0.044)	1.872 (0.018)
π_1	0.188 (0.014)	0.072 (0.010)	0.239 (0.058)
π_2	0.798 (0.015)	0.923 (0.011)	0.683 (0.092)

Dynamic Power Law Estimates: Exponent Series

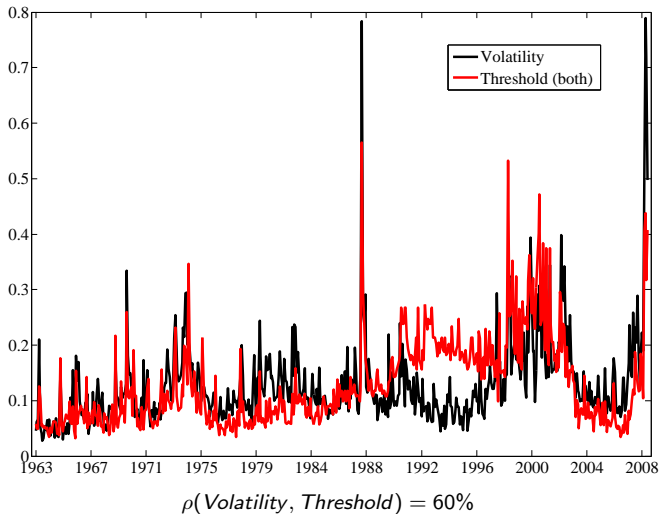


Dynamic Power Law Estimates: Exponent Series



$\rho(\text{Lower Exponent}, \log P/D) = -15\%$, $\rho(\text{Upper Exponent}, \log P/D) = -14\%$

Dynamic Power Law Estimates: Threshold



Testing Model Implications: Predicting Stock Returns

- Theory suggests increases in tail risk forecast increases in excess returns
- Predictive regressions of excess returns on aggregate market over short (one month) and long (up to five year) horizons
- Compare against common alternatives (dividend-price ratio, term spread, etc.)
- Robustness

Testing Model Implications: Predicting Stock Returns

Univariate Prediction

	One month horizon			One year horizon			Five year horizon		
	Coef.	t-stat	R ²	Coef.	t-stat	R ²	Coef.	t-stat	R ²
Tail Risk (-ζ lower)	6.70	2.9	0.016	4.44	2.1	0.069	5.02	2.3	0.272
Book-to-market	0.81	0.3	0.000	1.76	0.7	0.011	1.15	0.4	0.013
Cross section premium	5.53	2.4	0.010	-4.19	1.8	0.061	-4.28	1.9	0.179
Default return spread	1.73	0.7	0.001	-0.08	0.1	0.000	-0.12	0.6	0.000
Default yield spread	4.65	2.0	0.008	2.26	0.9	0.018	2.45	1.3	0.063
Dividend payout ratio	-0.27	0.1	0.000	0.88	0.4	0.003	1.90	0.7	0.028
Dividend price ratio	2.55	1.0	0.002	3.19	1.2	0.036	3.15	1.4	0.092
Dividend yield	2.65	1.1	0.003	3.18	1.2	0.036	3.10	1.4	0.089
Earnings price ratio	2.80	1.1	0.003	2.86	1.1	0.029	2.17	0.9	0.049
Inflation	-6.72	2.6	0.016	-1.99	1.1	0.014	-0.21	0.1	0.000
Long term return	5.35	2.3	0.010	2.21	3.0	0.017	1.18	2.5	0.014
Long term yield	-0.70	0.3	0.000	1.50	0.6	0.008	3.93	2.0	0.151
Net equity expansion	-4.34	2.1	0.007	-2.11	0.9	0.016	-1.04	0.5	0.010
Stock volatility	0.15	0.1	0.000	0.44	0.2	0.001	-0.03	0.0	0.000
Term Spread	4.49	2.0	0.007	3.95	1.9	0.056	4.00	1.8	0.159
Treasury bill rate	-3.08	1.3	0.003	-0.81	0.3	0.002	1.26	0.6	0.016
Variance risk premium	9.73	3.3	0.035	4.34	3.1	0.038	-4.64	2.9	0.090

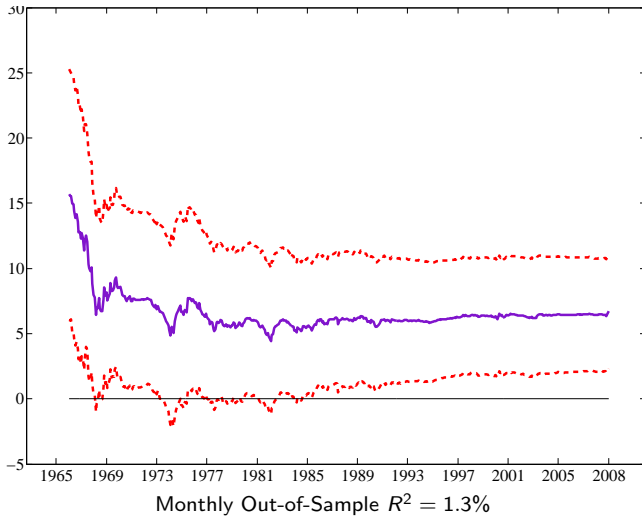
Testing Model Implications: Predicting Stock Returns

Bivariate Prediction

	One month horizon					One year horizon					Five year horizon				
	Coef.	t	Tail Coef.	Tail t	R ²	Coef.	t	Tail Coef.	Tail t	R ²	Coef.	t	Tail Coef.	Tail t	R ²
Book-to-market	0.73	0.3	6.47	2.8	0.015	1.70	0.7	4.41	2.1	0.080	1.05	0.4	5.04	2.3	0.283
Cross section premium	16.9	4.7	16.7	4.6	0.070	-2.17	0.9	2.97	1.3	0.081	-1.45	0.9	4.15	2.0	0.286
Default return spread	1.45	0.6	6.42	2.8	0.016	-0.28	0.5	4.44	2.1	0.070	-0.32	1.4	5.07	2.3	0.273
Default yield spread	3.36	1.5	5.72	2.5	0.019	1.33	0.5	4.13	2.0	0.076	1.35	0.7	4.75	2.0	0.290
Dividend payout ratio	0.49	0.2	6.54	2.8	0.015	1.42	0.7	4.59	2.2	0.077	2.67	1.2	5.34	2.3	0.325
Dividend price ratio	1.80	0.7	6.27	2.7	0.016	2.70	1.1	4.11	2.0	0.096	2.44	1.2	4.74	2.2	0.325
Dividend yield	1.78	0.7	6.23	2.7	0.016	2.61	1.0	4.06	1.9	0.094	2.29	1.2	4.72	2.1	0.319
Earnings price ratio	1.66	0.6	6.18	2.6	0.016	2.11	0.8	4.04	1.9	0.085	1.25	0.5	4.83	2.1	0.287
Inflation	-6.31	2.5	6.05	2.6	0.030	-1.70	1.0	4.31	2.0	0.080	0.17	0.1	5.07	2.3	0.272
Long term return	4.56	2.0	5.86	2.5	0.023	1.64	2.5	4.21	2.0	0.079	0.44	1.4	5.00	2.3	0.273
Long term yield	-3.45	1.5	7.71	3.2	0.019	-0.10	0.0	4.46	2.0	0.070	2.24	1.3	4.24	2.0	0.313
Net equity expansion	-2.34	1.1	5.66	2.3	0.017	-0.63	0.3	4.21	2.0	0.071	1.08	0.6	5.44	2.2	0.281
Stock volatility	1.02	0.4	6.62	2.9	0.016	1.04	0.6	4.56	2.1	0.074	0.62	0.6	5.14	2.4	0.275
Term Spread	2.46	1.0	5.59	2.3	0.017	2.70	1.3	3.45	1.6	0.092	2.36	1.3	4.20	2.1	0.319
Treasury bill rate	-3.92	1.7	6.96	3.0	0.021	-1.37	0.5	4.59	2.2	0.077	0.57	0.3	4.99	2.3	0.275
Variance risk premium	9.19	2.9	6.73	2.0	0.052	4.63	3.3	7.42	2.6	0.228	-3.90	2.9	5.45	1.9	0.313

Testing Model Implications: Predicting Stock Returns

Out-of-Sample Prediction (Lower Tail)



Testing Model Implications: The Cross Section of Returns

- Theory predicts
 - ① Differential exposure to tail risk state variable implies cross-sectional difference in expected returns
 - ② Negative price of tail risk: assets with high beta on tail risk have hedge value
- Test for cross-sectional relation between individual asset/portfolio return tails and returns
 - ① Returns on tail risk beta-sorted portfolios
 - ② Fama-MacBeth tests
 - ③ Robustness to alternative characteristics

Testing Model Implications: The Cross Section of Returns

Tail Beta-Sorted Portfolios: NYSE/AMEX/NASDAQ Stocks

		Tail Risk Beta					Diff. (5-1)	Diff. t-stat
		Low 1	2	3	4	High 5		
Panel A: Tail Risk Beta Only								
All		6.40	7.13	6.23	4.44	0.36	-6.03	2.55
Panel B: Market Beta / Tail Risk Beta								
Low β_{MKT}	1	6.71	7.41	7.11	6.40	3.65	-3.06	1.76
	2	5.91	6.19	5.97	4.50	2.27	-3.64	2.01
	3	4.32	5.12	4.34	3.25	0.44	-3.88	2.08
	4	2.54	3.36	2.53	1.06	-1.01	-3.55	1.87
High β_{MKT}	5	-0.02	1.78	-0.35	-1.57	-4.45	-4.43	2.20
Panel C: Market Equity / Tail Risk Beta								
Small	1	10.16	9.27	10.15	10.67	13.98	3.82	1.72
	2	2.76	4.48	3.46	0.51	-4.33	-7.10	2.81
	3	4.81	5.81	4.99	0.86	-5.32	-10.13	4.18
	4	6.71	7.72	6.45	4.79	-2.18	-8.89	3.69
Big	5	6.76	6.82	6.80	5.68	1.43	-5.33	2.31
Panel D: Book-to-Market / Tail Risk Beta								
Growth	1	5.75	6.16	5.23	3.05	-1.88	-7.63	3.07
	2	7.50	7.34	7.07	5.23	1.94	-5.56	2.37
	3	9.14	8.78	8.11	7.22	4.71	-4.43	1.97
	4	10.35	9.93	9.00	8.98	7.11	-3.24	1.52
Value	5	11.09	10.66	11.64	12.01	14.06	2.96	1.42

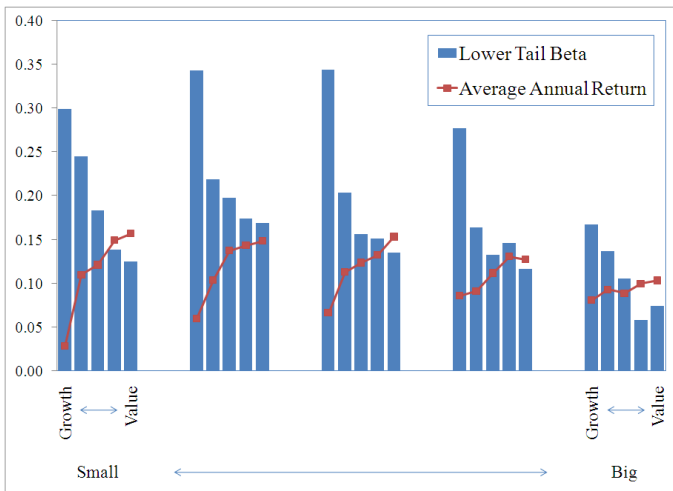
Testing Model Implications: The Cross Section of Returns

Stage 2 Fama-MacBeth Results: NYSE/AMEX/NASDAQ Stocks

$-\zeta$ (both)	-5.303 2.5			-5.582 2.3			-4.399 2.4		
$-\zeta$ (lower)		-6.197 2.9			-5.782 2.7			-5.586 2.9	
$-\zeta$ (upper)			-0.112 0.1			-0.005 0.0			-1.617 1.2
R. Vol.				6.001 2.9	6.309 3.1	6.863 3.6	5.209 2.8	4.663 2.6	4.978 3.0
R_{MKT}^e							-1.249 0.7	-1.150 0.7	-1.087 0.6
SMB							-3.787 2.4	-3.596 2.3	-3.583 2.3
HML							-0.940 0.6	-0.930 0.6	-0.588 0.4
Intercept	2.420 1.1	2.268 1.0	-0.645 -0.2	4.082 2.2	4.023 2.2	4.561 2.3	5.264 6.0	4.883 5.6	4.647 5.3
R^2	0.030	0.029	0.020	0.053	0.056	0.031	0.143	0.151	0.135

Hedging Tail Risk

Tail Risk Betas: 25 Size/BM Ptf and VIX



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Conclusions

- Derive link between return tails and risk premia in an affine pricing framework with tail risk
 - Present new methodology for capturing dynamic extreme risk in the economy
 - Identify substantial time variation in tails
 - Empirics consistent with predictions of structural model
 - ① Large variation in tail risk over time
 - ② Tail exponent forecasts excess market returns
 - ③ Associated with large cross-sectional differences in average returns
- ★
- What next?
 - ① Unified pricing with other asset classes (options and credit)