

Forced Liquidations, Fire Sales, and the Cost of Illiquidity

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“Vee try to tell them dat our problem was not a solfency problem but a likvity problem, but they did not agree.” – A drunken senior staffer at the Central Bank of Iceland—As told in *Boomerang*, Lewis, (2011).

“JPMorgan Chase & Co. (JPM)’s announcement that an internal inquiry may show “intent” to misprice trades in a unit that lost \$5.8 billion may help a U.S. investigation while putting distance between management and any wrongdoers. ‘Emails, voice tapes and other documents, supplemented by interviews’ were ‘suggestive of trader intent not to mark positions where they believed they could execute,’ the bank said in a presentation yesterday as it reported net income fell 9 percent to \$4.96 billion. ‘Traders may have been seeking to avoid showing full amount of losses,’ the bank said, noting management had concerns about the integrity of the prices used”—As reported by Leising and Hurtado in *Bloomberg News*, (2012).

1. Introduction

Institutional investors seeking diversification often build portfolios using collections of securities with widely varying characteristics. To help achieve diversification, investors generally use the common “currencies” of reported return, volatility, and correlation to construct or optimize their portfolio. As a result, investors using this approach are often drawn to investment opportunities that appear to exhibit diversifying properties simply because of the limited price discovery associated with those investments. Such opportunities are often relatively illiquid when compared to traditional investments like large cap stocks or sovereign debt, and investors frequently take for granted that they receive a “liquidity premium” that compensates them fairly for the lack of liquidity.¹ A variety of approaches have been proposed to incorporate liquidity into the portfolio optimization process: Seigel (2008), and Leibowitz and Bova (2009), develop methods for institutional investors to explicitly take liquidity into account when determining optimal asset weights; Ang, *et al.* (2011) characterize an investor’s optimal liquidity policy when there are frictions in the market; Lo, *et al.* (2003) add liquidity as an additional constraint in a mean–variance optimization; and Kinlaw, *et al.* (2013) incorporate liquidity as a shadow allocation to the portfolio.

The most common way to measure illiquidity in investments like hedge funds or private equity is serial correlation² in the reported return series of the investment, because such serial correlation is frequently viewed as the result of price smoothing caused by exposure to less liquid securities or investments. More sophisticated investors may adjust the return data by taking into consideration observed serial correlation in order to decode the true volatility of the portfolio; they thereby correct both the volatility and the risk adjusted performance of the investment (Scholes and Williams (1977); Geltner(1993); Getmansky, *et al.* (2004);

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¹Sparrow and Ilijanic (2010) quantify the value of liquidity in a trading context, and Amihud, *et al.* (2005) provide a literature survey of the theoretical ways liquidity effects asset price.

²It went up (down) last period; so odds are it will go up (down) again this period—also referred to as autocorrelation. See *e.g.* Scholes and Williams (1977).

Bollen and Pool (2008); Anson (2010 and 2013)). Simply adjusting for serial correlation however fails to measure or capture the core risk and cost of illiquidity: forced liquidations and “fire sales”.

Forced liquidations typically occur when illiquid portfolios become overvalued relative to their true market value (a relevant, timely valuation) and the reported valuation is no longer credible. Fire sales, or rapid sales of assets which depress prices, result when managers attempt to sell illiquid instruments or investments quickly. When these sales occur during significant adverse movements in the broader market with an associated high demand for general liquidity, the price depression is exacerbated. Most importantly, forced liquidations and fire sales often occur without warning, because they are precipitated by factors outside the investor’s control.

Many investors do not understand the true risk or cost of illiquidity until a forced liquidation or fire sale actually occurs—unfortunately, too late to help them. But by applying the barrier option pricing framework presented in this paper to the expected return of an illiquid investment, investors can often know the probable cost of illiquidity in advance. The method described here allows investors to use a combination of market data and experience combined in a consistent, analytically rigorous framework to derive a fair value estimate of the cost of illiquidity.

2. Causes of Illiquidity

When it comes to liquidity, not all securities are created equal. Ask any experienced portfolio manager to describe the agony associated with exiting a losing investment in the face of an illiquid, declining market—there is a qualitative difference and a real cost to exiting an illiquid portfolio compared with that of a liquid portfolio. Yet, in spite of the “lessons” learned during the global financial crisis of 2007–2008, many institutional investments remain illiquid, and this illiquidity may be exacerbated during an extended low interest rate environment with low volatility, because asset managers are seeking higher returns.

One can think of the primary cause of illiquidity as a mismatch between the funding of the underlying investment and the horizon over which the investment can be sold. Further, the greater the leverage employed in the investment, the more likely it is that illiquidity will have a deleterious effect on the value of the investment in a declining market. Positions in exchange–traded securities can generally be sold very quickly, although the seller of a large holding may experience significant price decline. In contrast, for an investment in real estate, even if the investor is willing to sell at a steep discount, it may still take a long time to find a buyer. When investments are supported through the use of any form of short–term leverage³ or transactions that have embedded liquidity puts,⁴ there exists the potential for a funding mismatch and, therefore, illiquidity.

Additional factors can increase the illiquidity of any particular investment. Contractual terms like redemption notice periods, lockups, or gates have liquidity costs that have been explored by Ang and Bollen (2010). They estimate that a three month redemption notice period, combined with a two year lockup for hedge funds, costs investors 1.5% of their initial investment and, if a gate is imposed, there is an additional cost which can exceed 10.0%. Other factors—so called “network” factors—may not be as readily apparent. The use of common service providers (custodians, prime brokers, securities lending counterparties, or pricing providers); common investors such as fund–of–funds or large institutions (Battacharya, *et al.* (2013)); or strategies which turn out to be correlated in unanticipated ways (Boyson, *et al.* (2010)) can create unforeseen illiquidity. In general, factors which cause implicit linkages may serve to create or increase illiquidity for a particular investment or portfolio.

Investors in collective investment vehicles (such as hedge funds or private equity) are also subject to the actions of other investors in the same (or similar) portfolios. In some circumstances, if even a single large investor decides to exit an investment, it can cause managers to sell assets to meet redemptions (Gennaioli, *et al.* (2012)). A high degree of leverage in the portfolio can also result in a rapid decrease in the value of the investment vehicle and thereby cause other investors to react. The economic advantage to being an early redeemer if a portfolio or asset is under stress is well known (see *e.g.* Mitchell, *et al.* (2007) or Chen,

³Examples of short–term leverage may include margin borrowing (see *e.g.* Garleanu and Pedersen (2009) or Brunermeier and Pedersen (2009)) or the use of futures, options, or swaps (see *e.g.* Office of Financial Research (2013)).

⁴Transactions which contain contractual obligations requiring liquidity upon demand, such as securities lending or repo transactions (see *e.g.* Keane (2013) or Office of Financial Research (2013)).

et al. (2010)), since slower investors end up holding shares of an increasingly less liquid portfolio (Manconi, *et al.* (2012)). For that reason, it has been shown that sophisticated investors withdraw much more quickly when there are questions associated with the liquidity of an investment (Schmidt, *et al.* (2013)).⁵

3. Liquidity and Reality

Consider first the nature of returns associated with trading and pricing a liquid security (or portfolio of securities) versus that of an illiquid security. For liquid securities there exist virtually continuous, objective, and reliable price discovery; generally even large transactions do not suffer a significant increase in cost upon liquidation. In contrast, illiquid securities, may trade only by appointment, at infrequent intervals, and without reliable, objective, public reporting.⁶ Sellers of illiquid securities cannot be certain about the market value of their holdings, but sellers know (or should know) that a security's sale price is likely to vary depending on the amount to be sold, the need to effect the transaction quickly, and price pressure associated with other sellers in the market.⁷ All else equal, if a seller needs to sell a large amount in a short time then the price received can decline dramatically.

This lack of a liquid, transparent pricing mechanism tends to produce relatively predictable behavior by the managers of less liquid portfolios. For example, it has been documented that portfolios of less liquid securities exhibit a high degree of positive serial correlation.⁸ In these cases, a significant proportion of the return in the current period can be statistically explained by the return in the prior period. This serial correlation is, of course, less likely to be the product of a wide-scale exploitable market anomaly, than the result of the valuation practices of the managers of such less liquid portfolios (it may be useful to think of private equity or certain hedge fund investments). Absent objective pricing, these portfolios tend to get marked (priced) "conservatively". In other words, prices are adjusted (and reported) by some proportion of the perceived difference between the point where they were marked during the prior period and the point at which they are believed to be "tradable" today. This approach does not necessarily imply nefarious behavior on the part of the manager,⁹ but can represent a simple Bayesian updating rule, using a partial adjustment or adaptive expectations approach, which is rational in that it minimizes the mean squared difference between the estimated value and the market value (Quan and Quigley (1991)).

Over time, assuming markets move both up and down, the reported value (reported asset value or R) of the illiquid asset will be a relatively unbiased predictor of the true¹⁰ "tradable" value (true asset value or N).¹¹ However, to the extent that a portfolio is "conservatively marked," the reported returns will be autocorrelated, in other words, the current or observed return (r_t^o) will be determined, in part, by the prior period reported return (r_{t-1}^o).

Why is this so? Consider Equation (1) which describes a typical "conservative" pricing strategy. The reported return for the illiquid portfolio in the current period (r_t^o) is determined by adjusting the change in portfolio value by some proportion, λ (the reporting adjustment), of the difference between the prior period reported value R_{t-1} and N_t , the true, tradable value today.

$$r_t^o = \frac{\lambda(N_t - R_{t-1})}{R_{t-1}} \quad (1)$$

We assume that the portfolio N_t , follows a discrete Brownian motion:

$$N_t - N_{t-1} = N_{t-1}\mu\Delta t + N_{t-1}\sigma\varepsilon_t\sqrt{\Delta t} \quad (2)$$

⁵Hill (2009) argues that options with long volatility exposures hedge some liquidity risk; Bhaduri, *et al.* (2007) apply this concept in a number of hedging strategies; and Golts and Kritzman (2010) suggest that investors can protect for unscheduled capital calls by purchasing liquidity options which pay the investor in those states of the world when such calls are likely.

⁶This is often the reason for low measured correlation and volatility.

⁷This observation has been formalized by Acerbi and Scandolo (2008).

⁸See *e.g.* Weisman (2003) or Getmansky, *et al.* (2004).

⁹Nor does it preclude such behavior.

¹⁰"True" should be thought of in a generic sense as being some prudent central value for the price that would be obtained during a given period if one wished to buy or sell the illiquid asset or portfolio.

¹¹This paper uses the terms "asset" and "portfolio" interchangeably—one either has a single illiquid asset (*e.g.* an investment in private equity or a hedge fund) or a portfolio which contains illiquid assets (*e.g.* the even more illiquid portion of a hedge fund).

Here, the true value of the portfolio in the current period N_t is equal to its true value in the prior period N_{t-1} multiplied by the trend rate of return (μ), and the one-period time step (Δt), plus the assumed volatility of the process (σ) multiplied by the product of ε_t (a standard normal random variable) and the square root of the one-period time step ($\sqrt{\Delta t}$). Thus, the change in value in any given period is the result of a combination of a trend and a random shock.

Substituting Equation (2) into Equation (1) and rearranging results in Equation (3), an expanded representation of the observed rate of return over a period of length (Δt):

$$r_t^o = \frac{\lambda(N_{t-1}(1 + \mu\Delta t) - R_{t-1}) + \lambda(N_{t-1}\sigma\varepsilon_t\sqrt{\Delta t})}{R_{t-1}} \quad (3)$$

Finally, taking the expectation of r_t^o yields:¹²

$$E[r_t^o] = \frac{\lambda(N_{t-1}(1 + m\Delta t) - R_{t-1})}{R_{t-1}} \quad (4)$$

As a consequence, the proportion of the expected value for period Δt explained by the actual value of the portfolio is λ . Recognize that although N_t and R_t have the potential for significant differences over time, $(1 - \lambda)$ of the observed return will be explained by the prior period's actual difference between N_{t-1} and R_{t-1} . The effect of this relationship is to induce first order serial correlation $\rho_{s(1)}^o$ in the observed return series. This first order serial correlation is proportional to $(1 - \lambda)$ even though the error term that helps to drive the true return of the portfolio may be independent through time. For a quick and dirty estimate of the extent to which a portfolio manager is "conservatively marking" a portfolio (understating the change in its real value), one can simply calculate the first order serial correlation and subtract it from 1 to yield an estimate of the proportion (λ) of the true change in the portfolio value that is being reported by the manager.¹³ As noted, even with a process that systematically under-adjusts for changes in valuation from time period to time period, the reported asset value may still be a relatively unbiased representation of the true value. Given that, the question becomes whether this misreporting is merely a benign understatement of the true volatility of the portfolio, which can be ignored, or whether it has a real cost to the investor.

4. The Barrier Option Framework

To answer this question we begin by examining the dynamics of how the two related processes (N_t and R_t) evolve over time. Figure (1) illustrates how R_t might track N_t over a randomly generated period of 60 time intervals (for this simulation assume that $\mu = 0.05$, $\sigma = 0.25$, and $\lambda = 0.25$). The "conservative" reporting process (in red) tends to "smooth" the valuation of the portfolio through time and, as expected, exhibits less price volatility than the actual, underlying series (in blue).

¹²The expectation is obtained by applying Ito's lemma, where $m = \mu - \frac{\sigma^2}{2}$ and μ is the arithmetic mean return.

¹³This relationship is explored in Appendix II.

Figure 1: Hypothetical Growth of \$100

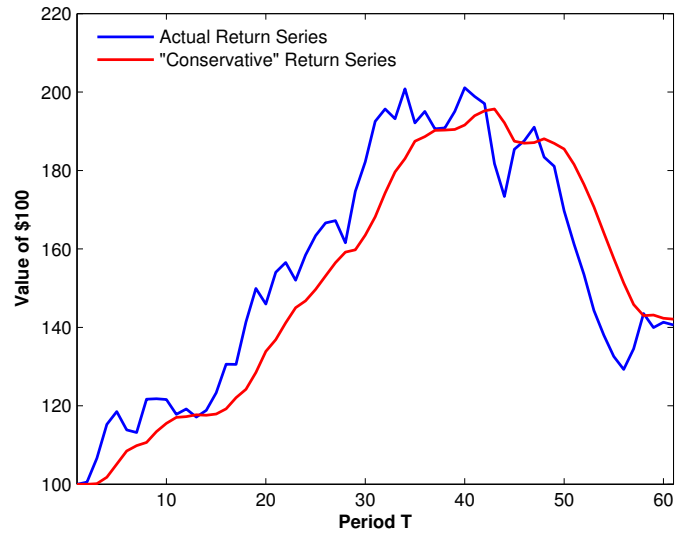
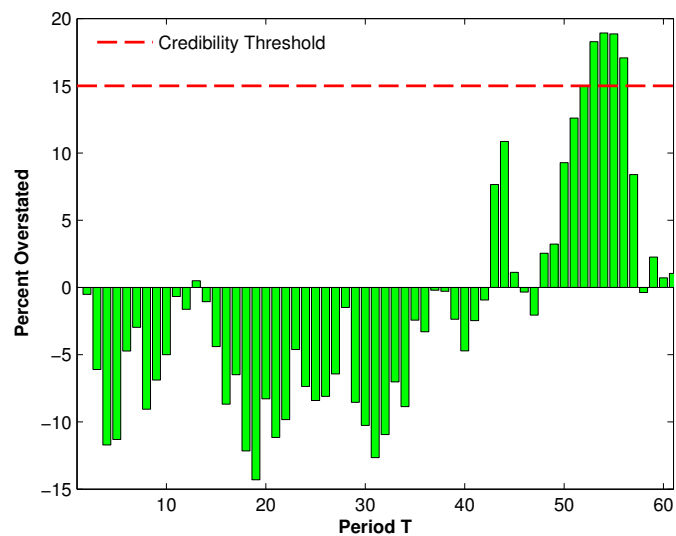


Figure (2) depicts the individual period differences between R_t and N_t , where values above zero correspond to periods when the manager is over-valuing the portfolio relative to its true value. The manager’s “conservative” approach to valuing the portfolio results in periods of both significant under- and overvaluation. From a practical standpoint for an investor, undervaluation tends to be less important than overvaluation since having an investment which is worth more than its stated value is rarely harmful. Undervaluation is also rarely a concern for third parties providing financing to support the portfolio, since those parties are over-collateralized. But parties providing financing are usually quite interested in overvaluation. For example, a prime broker extending credit to finance the portfolio positions will want to ensure that the manager’s appraisal of the portfolio’s value doesn’t exceed some rational tradable value by more than a “reasonable” margin, since that value serves as collateral for the financing.

Figure 2: “Conservative” less Actual Returns



This reasonable margin of overvaluation may be referred to as the “credibility threshold” (L). Figure (2)

sets the threshold at 15.0%. When a manager exceeds the credibility threshold there is often a response by interested parties; prime brokers tend to take action promptly, but the response can be slower with larger, more bureaucratic organizations like institutional investors, particularly when financial reporting is delayed (as is the case for hedge funds). Regardless of where it arises, a breach of the credibility threshold is likely to trigger forced behavior by the manager, in other words, the manager will be required to sell some or all of the illiquid portfolio in a relatively short time, typically in a descending or “thin” market. The single period loss that occurs thus consists of two components. The first is a loss governed by $(R_t - N_t)$, the extent to which the portfolio was overvalued. The second is a liquidation penalty (P) associated with a fire sale of the illiquid portfolio in a (typically) descending market. This liquidation penalty increases when the portfolio contains significant leverage since it is likely that more of the portfolio will need to be liquidated. Large, single period losses of this type are relatively common in financial markets and tend to be larger than losses estimated through the use of conventional and highly data-dependent methodologies such as value at risk (VaR) or expected shortfall (CVaR). However, this paper suggests that these losses are somewhat predictable, and that, by formalizing the basic structural dynamics described above, it is possible to develop an objective framework for analyzing the cost of illiquidity.

When the credibility barrier is breached and a manager is required to liquidate positions as outlined above, one can model the associated cost as an up-and-in barrier option on the path of reported valuation of the portfolio.¹⁴ When the path of reported value is overvalued and exceeds the threshold (barrier), the option “pays” (the “payment” is negative and represents a loss to the investor) the sum of two components, the amount by which the value of the portfolio was overstated and the additional loss associated with the forced liquidation. If the investor has a set of prior beliefs about: (a) the return and volatility characteristics of the portfolio (based on the observed mean, standard deviation, and serial correlation)¹⁵; (b) the conditions that will elicit a forced sale of the portfolio (*i.e.* a realistic estimate of the credibility threshold); and (c) the liquidation cost of being forced to sell a relatively illiquid portfolio during a stressful market period (an estimate of the liquidation penalty), it is relatively straightforward to price the “option” using Monte Carlo techniques.¹⁶

First, simulate the “true” value of the portfolio using a discrete Brownian motion, a function of the observed volatility, and trend rate of return, and its associated estimated valuation lag, λ (which is based on one minus the observed first order serial correlation, $(1 - \rho_{s(1)}^o)$).¹⁷ Then calculate the individual period differences between the two processes, and, when the difference exceeds the assumed credibility threshold, apply a payout equal to the difference between the two series at the time of the breach plus the assumed transaction penalty associated with a forced liquidation. Do this 100,000 times, calculating the net present value of whatever payout occurs for every one-year path, note that many paths will have no associated penalty. Finally, calculate the mean of the resulting payoffs (including the zero valued payoffs).

The option model is structured as a one year option so that the price translates as a “haircut” to the reported annualized rate of return associated with the investment. The price of the option is the *de facto* price of the risk assumed by investing in the less liquid portfolio because the “conservatively-valued” portfolio may become significantly overvalued and thereby force a sudden, expensive liquidation. To arrive at the liquidity-adjusted expected return of the portfolio, one simply subtracts the dollar value of the option from the expected return of the portfolio (*i.e.* if the option is valued at \$2.00 and the expected return of the

¹⁴We are not the first researchers to apply option theory to the issue of liquidity: Chaffe (1993) uses Black–Scholes option pricing to value illiquidity in private company valuations; Longstaff (1995) uses risk-neutral valuation to develop an upper bound on the value of marketability, and Golts and Kritzman (2010) use a cliquet option to hedge unexpected capital calls.

¹⁵Given that these are derived from returns reported by the manager, it is important to also use additional, independent sources to validate these parameters or the model may produce misleading results. See the discussion regarding the Bear Stearns fund in Section 6.

¹⁶Technically, the option is being given to the manager of the illiquid portfolio by the investor since the manager generally benefits from holding illiquid assets through an asymmetric compensation structure (*i.e.* the manager shares gains on the upside, but does not share investors’ losses on the downside). See Keane (2013) and Huang, *et al.* (2011).

¹⁷To generate paths for the “true” value of the portfolio, it is necessary to adjust the observed volatility for the serial correlation. We use the methodology developed by Geltner (1993):

$$\sigma = \sqrt{\left(\frac{1 - \rho_{s(1)}^o}{(1 - \rho_{s(1)}^o)^2}\right) \sigma_o^2}$$

where σ is the adjusted volatility, σ_o is the observed volatility, and $\rho_{s(1)}^o$ is the observed serial correlation.

investment is 10.00%, the adjusted return would be 8.00%).

5. An Example

To illustrate this approach, our base case is a portfolio with mean expected return, μ , of 6%; volatility, σ , of 12%; with a riskless rate in the market of 2%, all on an annualized basis.¹⁸ The one year barrier option is valued on an initial \$100 portfolio priced every week (52 times per year). When varying other parameters, we fix λ (the reporting adjustment) at 25%, L (the credibility threshold) at 15%, and P (the liquidation penalty) at 25%.¹⁹ The evaluation of the option price is always based on 100,000 Monte Carlo simulations.²⁰

Figure 3: Option Price as Function of λ and L

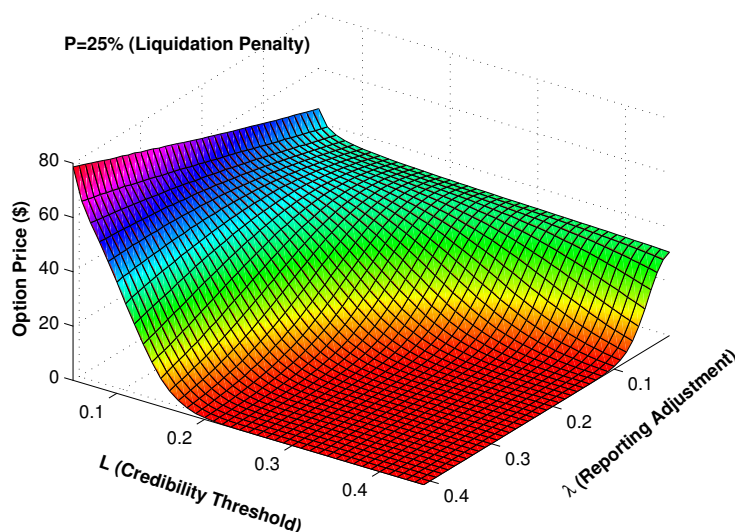


Figure (3) illustrates how the cost of the *de facto* option varies as a function of the reporting adjustment λ and credibility threshold L when the liquidation penalty is held constant at 25%. As λ decreases, *i.e.* as serial correlation in the portfolio increases, the option value increases. Since we have used a one year option on a \$100 portfolio, we can directly interpret the option value as an annual percentage cost to the investor for the illiquidity in the portfolio. For example, with a λ of 25%, a credibility threshold of 15%, and a liquidation penalty of 25% (our base case, approximately left of center in Figure (3)), the option has a value of \$15.54. This means that the investor should adjust the expected return for an investment in the portfolio by -15.54%. Since we have assumed an expected return of 6% for this portfolio, the illiquidity option that the investor is providing to the manager consumes all of the expected return from the investment; leaving an undesirable -9.54% liquidity-adjusted expected return.²¹ It is important to recognize that for all credibility thresholds below 20%, the value of the option is significant over a range of reporting adjustments (serial correlations).

Turning our attention to the influence of the credibility threshold, Figure (3) shows that as L increases (as the manager is given more latitude to overstate performance), the cost of the illiquidity option decreases. This result is expected since the likelihood that the credibility threshold will be breached goes down as the threshold increases. Significantly, no one investor controls this threshold. While a given investor may be

¹⁸These are approximately market values at the time of writing.

¹⁹We discuss the reasonableness of these values later in this Section.

²⁰Matlab code to price this option is included in Appendix I.

²¹Note that this has not been adjusted for the cost associated with notification periods, lockups, or gates as described in Section 2.

very lax (*i.e.* have a very high credibility threshold), it is the threshold of other investors or service providers that “controls”—it is the credibility threshold of the market that matters.²²

Figure (4) illustrates the relationship between the reporting adjustment and the liquidation penalty when the credibility threshold is constant at 15%. For higher reporting adjustment factors (lower serial correlations), the cost of the liquidity option is low. But for lower reporting adjustment factors (higher serial correlations), the cost of the liquidity option can increase dramatically. As might be expected, the cost of the option is monotonically increasing in the size of the liquidation penalty, P . The steepness of the cost function as the serial correlation increases underscores the cost of vanishing liquidity in a portfolio. As discussed in Section 2, when a portfolio is under stress, investors can be left with less and less liquid positions (resulting in higher and higher serial correlation). Figure (4) show that, in those cases, the cost of the illiquidity option can easily overwhelm the expected return of the portfolio.

Figure 4: Option Price as Function of λ and P

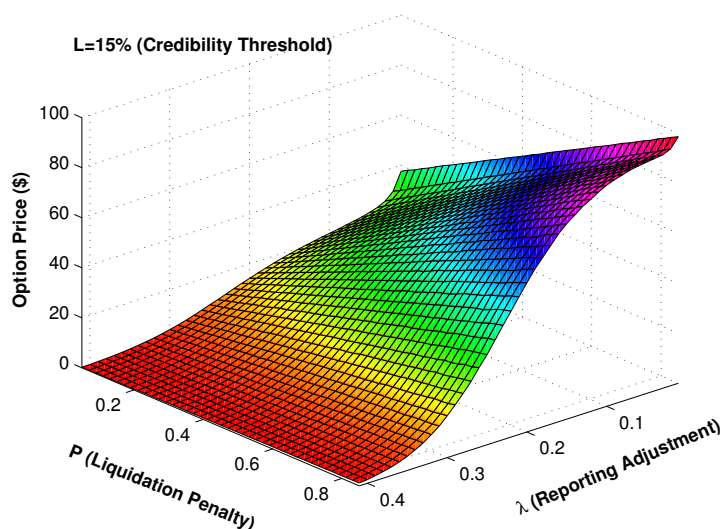
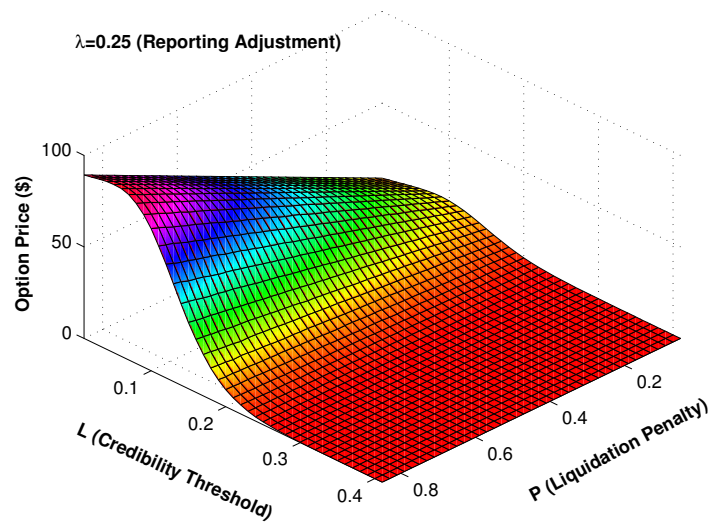


Figure (5) shows the relationship between the liquidation penalty and the credibility threshold when the reporting adjustment is held constant at 25%. As discussed above, when the credibility threshold is high (supervision is lax), the cost of the option is low. Otherwise, as the liquidation penalty increases, the cost of the option also increases. For sensible ranges of the credibility threshold (5 to 15%) and relatively high serial correlations (*e.g.* 75%), the entire range of liquidation penalties results in significant cost associated with the illiquidity option. While the value of the illiquidity option diminishes with lower serial correlation, the associated cost can still be important enough to investors to warrant estimation and consideration.

²²More precisely, it is the lowest credibility threshold of any investor or third-party who has the ability to trigger a fire sale.

Figure 5: Option Price as Function of P and L



It is also useful to consider the effect of the other parameters in the cost of the illiquidity option. When the riskless interest rate increases, the cost of the option goes down due to discounting—of course, the relative value of the investment should also be subject to the same discounting. When the number of pricing periods (*i.e.* the frequency at which the portfolio value is priced or marked) goes up, the cost of the option also goes down, because the market can more quickly identify any overvaluation and therefore act upon any breach of the credibility threshold earlier. For portfolios with higher volatility, the cost of the option increases since there is a greater likelihood of overvaluation, and for portfolios with higher expected returns, the cost of the option is lower since a greater expected return tends to offset the illiquidity option’s cost.

For the liquidation penalty, we have been assuming a base case value of 25%. Research by Ramadorai (2008), who analyzed transactions in the secondary market for hedge funds, found that, for those transactions involving fraud or collapse, the average discount to reported NAV was 49.6%, almost twice our base case value. As discussed earlier, as the liquidation penalty increases, the cost of the illiquidity option increases monotonically.

A few comments about this modeling approach. If one attempts to interpret the illiquidity option as the liquidity premium embedded in an investment, it appears to be much too high (approximately 15% in our base case). This is because the option approach used here does not price the liquidity premium investors usually think of, but rather represents the cost associated with the price smoothing of an illiquid investment which, when combined with a “triggering event”, results in an abrupt sale into a deteriorating market. The size of the illiquidity option is a function of the magnitude of manager mispricing and the cost of liquidating at an unfavorable price. The higher the degree of leverage employed in the underlying investment, the larger the cost associated with any liquidation in the event of a fire sale. Clearly, actions taken by the manager can mitigate the value of the illiquidity option. For example, the manager could vary the reporting adjustment (λ) dynamically rather than using the static approach modeled here.²³ This would allow the manager to control the degree of over- or under-valuation associated with the investment. In practice, we would expect that managers would utilize such an approach. A manager could also liquidate a portion of the portfolio as the credibility threshold is approached. However to modify behavior as the barrier is approached, the manager would need a well-formed expectation about the level of the credibility threshold, which may be difficult to obtain since the barrier is not pre-established and is set by actors outside the manager’s control. In addition, the incentives to cheat may increase as the barrier is approached. Many managers get into trouble when they make the decision to hide bad results in an attempt limit the scope for investor withdrawals. They often hope that the market will turn and bail them out of the situation. But most managers lack the flexibility

²³This type of dynamic or state-dependent choice could theoretically be embedded in an general equilibrium setting.

to alter their book. Attempting to institute a portfolio insurance strategy or to sell off part of the book will exacerbate the situation because the positions will now have an accurate mark. Managers and investors can generally live with a bit of variation around the “true” value of an illiquid investment, but if a manager decides that the credibility threshold is close, trying to modify holdings or positions may actually trigger a fire sale. Nonetheless, since we are not modeling an environment where the manager is making dynamic responses, we believe that the option value represents an upper bound on the cost of illiquidity.

Finally, turning to the assumptions associated with expected returns and volatility, JP Morgan Asset Management (2012), projects hedge fund expected returns in the range of 5 to 7% per year, with volatilities in the range of 7 to 13% per year—consistent with the assumptions of our base case. Importantly, JP Morgan estimates the expected returns for private equity at about 9% with a volatility of 34.25%. Since volatility serves only to increase the cost of the illiquidity option, only a high expected return (with reasonable levels for the other parameters) can serve to offset the cost of illiquidity in the portfolio. One could question whether middle, single digit returns are truly sufficient for investors to bear the cost of illiquidity that many hedge fund and private equity portfolios contain.

6. Pricing Liquidity in Alternative Investments

We are now in a position to consider several real world applications of the model and the implications for investors in less liquid portfolios. We begin by considering five well known hedge fund indices constructed by HFRI: (a) fixed income—convertible arbitrage, (b) distressed/restructuring, (c) multi—strategy, (d) fixed income—corporate, and (e) emerging markets Russia/Eastern Europe. Table 1 contains estimates for the first order serial correlation from the inception of each index, along with the reporting adjustment which would be implied in each case. Note that the values shown represent index level estimates, so it is reasonable to assume that there could be a significant degree of variability in the underlying funds which constitute the indices. As shown in Table 1, mean index serial correlations for most of these strategies lie in the 50 to 60% range, implying an adjustment factor less than 0.50, *i.e.* where managers, on average, reflect less than 50% of the true period-to-period change in the value of their portfolios. Depending on an investor’s assumptions concerning the liquidation penalty and the market credibility threshold, and estimates of the expected return and volatility, the adjustment to observed returns (based on the cost of the illiquidity option) could be quite significant. We would expect, however, for indices constructed with a large number of underlying funds that at the index or composite level, the over- and under-valuation of various managers would be “diversified” away²⁴ and that the liquidity option on an index should therefore be zero or close to zero. In fact, for the first four indices in Table 1, that is exactly what we find—the estimated option value is zero in each case. However, the Russia/Eastern Europe emerging market index, which reflects the lowest serial correlation in Table 1, has a monthly reported mean of 1.44% (17.30% annualized) and a monthly reported standard deviation of 7.69% (26.64% annualized). The liquidity option is priced²⁵ at 13.52, which reduces the seemingly large 17.30% return to only 3.78% per year, and demonstrates that serial correlation alone is not sufficient to determine whether or not there is a cost associated with lack of liquidity.

Table 1: First Order Serial Correlation of Select HFRI Hedge Fund Indices

HFRI Index	Serial Corr	“Estimated” λ
Fixed Income—Convertible Arbitrage	58%	0.42
Distressed/Restructuring	53%	0.47
Multi—Strategy	50%	0.50
Fixed Income—Corporate	48%	0.52
Emerging Markets: Russia/Eastern Europe	38%	0.62

The next application uses the Morningstar—CISDM hedge fund and CTA database which contains data for both alive and dead funds.²⁶ Eliminating CTAs and fund-of-funds, the initial sample contained data for

²⁴Geltner(1993) demonstrates a similar result in the case of real estate appraisals.

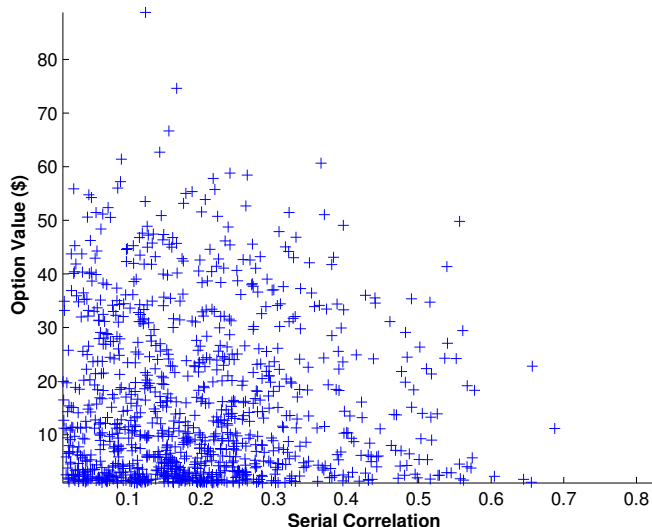
²⁵Based on the estimated mean, standard deviation, and serial correlation; a riskless rate of 2.00%, a credibility threshold of 15%, and a liquidation penalty of 25%.

²⁶Using both alive and dead funds minimizes survivorship bias in the analysis.

13,540 hedge funds. Selecting hedge funds for which there were at least 24 months of returns and those funds with a serial correlation greater than 0.01 results in a final sample of 3,554 hedge funds. Using this final sample, we computed the value of the liquidity option using a mean, standard deviation, and serial correlation estimated from the return series of each fund, less the last three months (*e.g.* if there were 30 months of data, we used the first 27 months to estimate the parameters to avoid including the period where the fund might fail); in all cases we used a risk free rate of 5%,²⁷ a credibility threshold of 15%, and a liquidation penalty of 25%. The mean annualized return for this sample of funds in the sample was 11.79%, the mean annualized standard deviation was 13.88%, and the mean serial correlation was 0.2032. The mean option value for these 3,554 hedge funds was 5.52, implying an actual liquidity-adjusted mean return of 6.27% on an annualized basis.²⁸

To focus on funds for which the liquidity option represented a real reduction in return, we selected hedge funds with positive average returns and an option value greater than 1.00 (that is, where the option reduces the expected return by 1.00% or more). This resulted in a final sample of 1,031 funds. For this sample, the mean annualized return for this sample of funds was 18.75%, the mean annualized standard deviation was 25.67%, and the mean serial correlation was 0.1959. The mean option value for these 1,031 hedge funds was 15.82, implying an actual liquidity-adjusted mean return of 2.93% on an annualized basis—a sharp contrast from the supposed 18.75%.²⁹ Narrowing the sample allows us to further explore the application of the liquidity option to real data. For example Figure (6) confirms the observation made earlier with the HFRI indices—serial correlation alone is not a proxy for the value of the liquidity option. And, as shown in Figure (7) serial correlation does not have a particularly strong relationship to maximum drawdown.

Figure 6: Option Value versus Serial Correlation

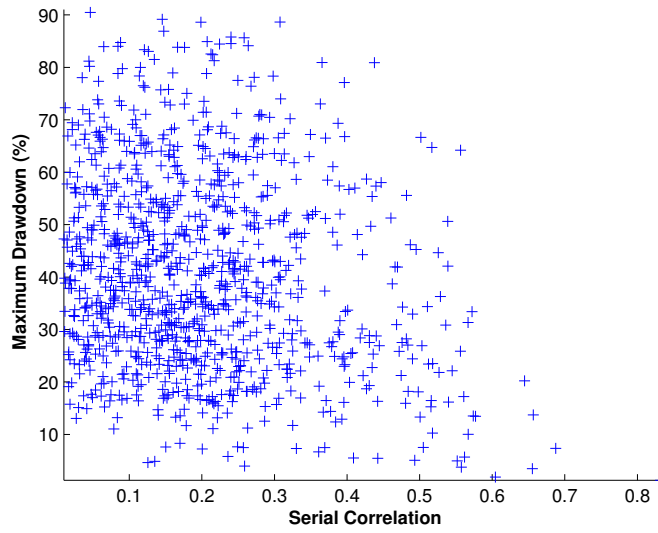


²⁷This is approximately the average risk free rate for the period.

²⁸1,357 of the funds had a zero option value.

²⁹Of course, there is a purposeful selection bias here.

Figure 7: Maximum Drawdown versus Serial Correlation



From Figure (8) we can tell that the mean return of the fund has only a weak relationship with the value of the liquidity option, and from Figure (9), that the mean return has no relationship with maximum drawdown.

Figure 8: Option Value versus Average Return

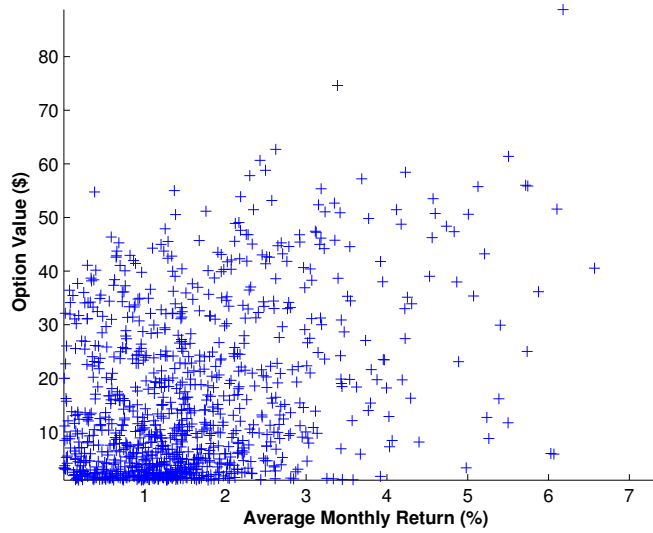
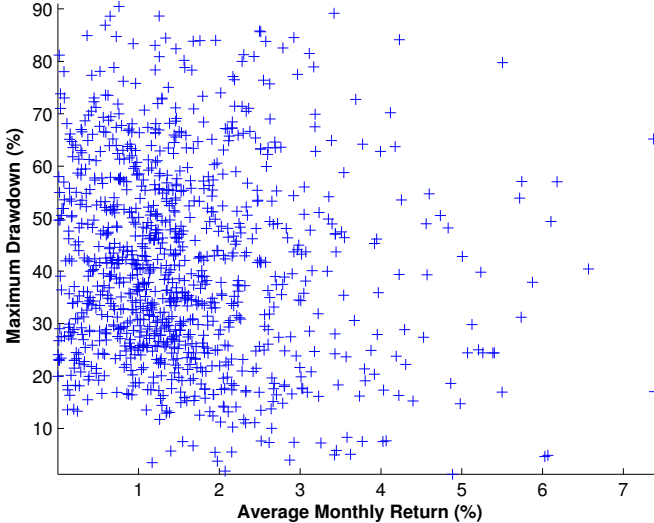


Figure 9: Maximum Drawdown verses Average Return



As we would expect from standard option pricing theory, the value of the liquidity option has an extremely strong relationship with the volatility, shown in Figure (10). However, Figure (11) demonstrates that there is little relationship between the maximum drawdown experienced in the the hedge fund sample and volatility, since the vast majority of drawdowns occur with volatilities (monthly standard deviations) less that 12.5%. So while volatility of returns is an important determinant of the value of the liquidity option, it appears to be unrelated to drawdown.

Figure 10: Option Value verses Volatility

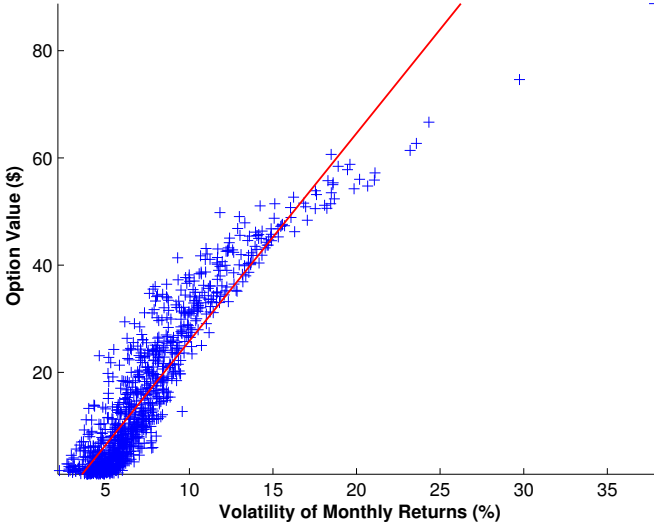
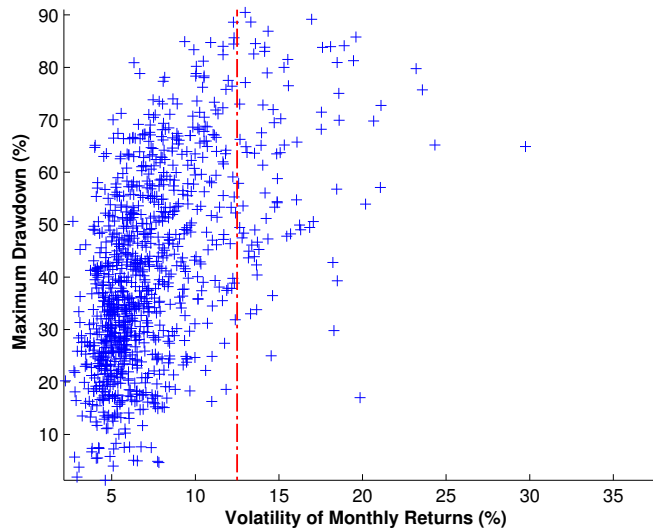
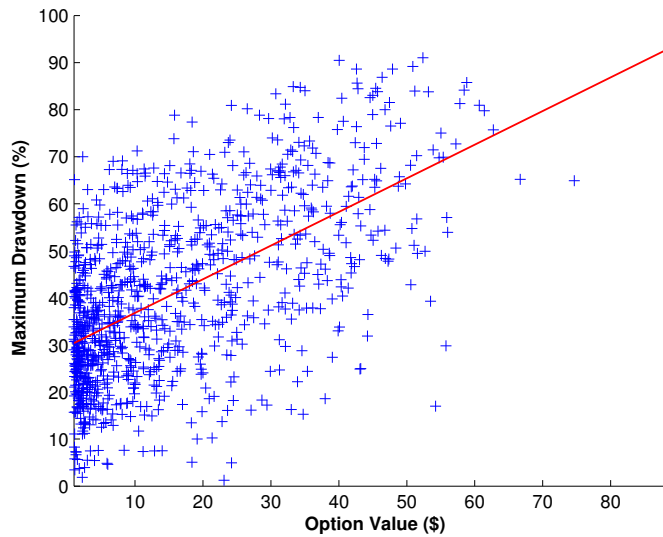


Figure 11: Maximum Drawdown versus Volatility



Contrast this with the strong relationship between drawdown and the option value shown in Figure (12). It is clear that higher drawdowns are associated with higher option values.

Figure 12: Maximum Drawdown versus Option Value



It is also informative to look at individual hedge funds and the pricing of their associated options. Table 2 shows the results for sixteen hedge funds drawn from our final sample, ordered by annualized return. These results underscore the general results described above. For example, the two funds with highest serial correlation, Alta Partners and Bear Stearns Asset Backed Securities, have relatively low valued liquidity options, whereas the liquidity options for Bridgewater Partners and Okumus Opportunity, which have relatively low serial correlation, represent significant reduction in the expected returns of the funds.

As a final example, consider the Bear Stearns High-Grade Structured Credit Strategies fund, whose marketing materials stated that the “fund seeks to generate total annual returns through ‘cash and carry’

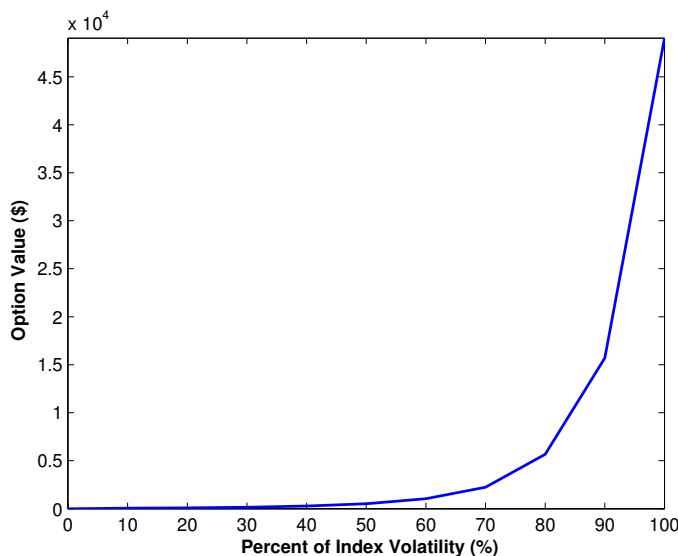
Table 2: Select Hedge Fund Performance and Adjusted Performance

Fund Name	Annualized Return	Serial Correlation	Option Value	Adjusted Annual Return
Deephaven Credit Opportunities	1.88%	0.42	\$1.32	0.56%
Bridgewater Partners	3.58%	0.20	\$8.21	-4.62%
Thames River European A (EUR)	6.76%	0.03	\$1.28	5.47%
Thames River Property Growth & Income (EUR)	7.24%	0.43	\$1.64	5.60%
Glenrock Global Partners (BVI), Inc.	7.42%	0.11	\$1.01	6.40%
Everest Capital Intl.	8.31%	0.27	\$9.34	-1.03%
Rocker Partners	9.42%	0.09	\$7.46	1.96%
Alta Partners L.P. (Onshore)	11.13%	0.64	\$1.67	9.46%
Rainbow Global High Yield (USD)	16.03%	0.28	\$14.31	1.71%
Marathon Emerging Markets	16.20%	0.23	\$20.05	-3.85%
Bear Stearns Asset Backed Securities LP	24.86%	0.60	\$2.24	22.61%
Okumus Opportunity A	27.53%	0.04	\$36.48	-8.95%
Viaticus	31.22%	0.05	\$33.03	-1.81%
Lancer Offshore	33.63%	0.16	\$9.68	23.95%
Galleon Omni Technology (B)	40.25%	0.13	\$45.76	-5.50%
Infinity Emerging Opportunities	74.16%	0.12	\$88.80	-14.64%

transactions and capital markets arbitrage.³⁰ It further stated that “the Fund generally invests in high quality floating rate structured finance securities.” The High-Grade Structured Credit Strategies fund is a poster child for a fire sale and rapid liquidation. As subprime mortgage delinquencies grew, the value of CDOs held by the fund dropped. The fund’s prime brokers asked for more cash collateral. The fund attempted to meet those collateral calls through liquidation of assets in a rapidly deteriorating market, but values fell quickly and collateral requirements rose rapidly, leading to eventual collapse. The fund failed in spite of an attempt to stabilize the fund value through the substantial injection of capital by Bear Stearns.

Prior to the collapse of the fund, the annualized mean return was 12.40% with an annualized standard deviation of 1.50% and a serial correlation of 0.6365. Straightforward pricing of the liquidity option produces a value of zero³¹—seemingly at odds with the approach advocated here. But notice that the reported standard deviation is extremely low, much lower for example than the standard deviation for the comparable HFRI index (the Fixed Income-Asset Backed Index) which is 4.03%. Figure (13) plots the value of the liquidity option for the Bear Stearns fund when the volatility of the fund is assumed to be a varying percentage of the volatility of the HFRI Fixed Income-Asset Backed Index. The reported volatility of the Bear Stearns fund is approximately 37% of the volatility of the index (which results in an option value essentially equal to zero), but as the assumed volatility of the fund approaches that of the index, the value of the liquidity option explodes exponentially (note that the option value on the y-axis is in tens of thousands). This example underscores the importance of having appropriate estimates for the parameters of the liquidity option model. In the case of the Bear Stearns fund, the reported volatility was significantly lower than what the market should have expected given the assets held by the fund. For this reason, it is important to have a “sanity” check on the reported values of a fund when computing the value of the liquidity option—artificially low volatility of returns or artificially high expected returns can make the analysis superfluous.

Figure 13: Option Value as Function of Index Volatility



7. Conclusion

Illiquidity, even in its milder forms, is not a benign condition that results merely in lower reported volatility for a portfolio. Underestimation of the cost of illiquidity during the portfolio construction process can produce significant distortions—understated volatility, understated correlations, overstated expected returns, and an over allocation to less liquid securities. One need only consider the predicament of JP Morgan as it was forcibly extricated from its overvalued “London Whale” trades during an abbreviated “career defining” one

³⁰There were two Bear Stearns funds following similar strategies, here will use only one as an example.

³¹Again, assuming a credibility threshold of 15%, a liquidation penalty of 25%, and a risk free rate of 5.0%.

month period, or for the investors in “quant” hedge funds in August of 2008 to appreciate how large these costs can be. To adjust properly for the embedded risks of illiquidity, we propose a straightforward barrier option pricing model which provides an objective framework for incorporating market data and investor experience; and thereby allows institutions to evaluate the true cost of illiquidity in their investments.

It is also important for investors to recognize that the liquidity of an investment is not necessarily constant through time. When a portfolio comes under funding stress, the easiest and quickest way to alleviate that stress is to sell the most liquid assets in the portfolio. But as liquid assets are sold, the portfolio becomes even more illiquid, generally over short time horizons—driving up, sometimes rapidly, the cost of the embedded illiquidity option.

As this paper has shown illiquidity in the market place has discernible costs. The barrier option pricing methodology described here can provide both a screen and a rigorous adjustment mechanism for evaluating investments which appear attractive, but may actually be unduly risky. Thus in the case of the Bear Stearns fund discussed above, a modest upward adjustment to the fund’s reported volatility revealed an embedded option cost that dwarfed the reported returns, because of the fund’s highly serially correlated returns. This paper’s rigorous framework for assessing liquidity provides the fund manager with the means to make an annualized adjustment to reported returns that corrects for illiquidity.

Appendix I: Matlab Code

```
% Illiquidity_Option.m

% From the paper "Forced Liquidations, Fire Sales, and the Cost of Illiquidity",
% by Richard R. Lindsey and Andrew W. Weisman

% Set basic parameters
S0 = 100; % Starting value of portfolio
mu = 0.05; % Expected return of the portfolio
sigma = 0.25; % Adjusted volatility of the portfolio using Geltner (1993) methodology
irate = 0.02; % Riskless interest rate
lambda = 0.25; % Reporting adjustment (1- first order serial correlation)
threshold = 0.10; % Estimate of market credibility threshold
penalty = 0.25; % Estimate of liquidation penalty
T = 1; % One year
NSteps = 52; % Number of pricing periods (52 weeks)
NRepl = 1000000; % Number of replications in the Monte-Carlo simulation
dt = T/NSteps;
index = NSteps+1;

% Pre-allocate and initialize
SPaths = zeros(NRepl, index); % SPaths are the actual values
SPaths(:,1) = S0;
LPaths = zeros(NRepl, index); % LPaths are the conservative values
LPaths(:,1) = S0;

% Monte-Carlo Simulation
nudt = (mu-0.5*sigma^2)*dt;
sidt = sigma*sqrt(dt);
for i = 1:NRepl
    for j = 1:NSteps
        SPaths(i,j+1) = SPaths(i,j)*exp(nudt + sidt*randn);
        LPaths(i,j+1) = LPaths(i,j)+lambda*(SPaths(i,j)-LPaths(i,j));
    end
end

% Over or understatement compared to credibility threshold
diff = ((LPaths-SPaths)./SPaths)*100;
[row,col] = find(diff >= threshold*100);
urows = unique(row);
firsttime = zeros(length(urows),1);
for i = 1:length(urows)
    firsttime(i) = find(diff(urows(i),:)>=threshold*100,1);
end

% Preallocate
LiqLoss = zeros(length(urows),1);
OverState = zeros(length(urows),1);
OptionVal = zeros(length(urows),1);

% Determining the cost of the illiquidity option
for i = 1:length(urows)
    LiqLoss(i) = SPaths(urows(i),firsttime(i))*penalty;
```

```
    OverState(i) = diff(urows(i),firsttime(i));
    OptionVal(i) = (LiqLoss(i) + OverState(i)) * exp(-irate*(firsttime(i)/NSteps));
end

CostOption = sum(OptionVal)/NRepl;
```

Appendix II:

The true value of the fund at time t is given by:

$$N_t = N_{t-1}(1 + r_t)$$

rearranging to get the true return r_t :

$$r_t = \frac{N_t}{N_{t-1}} - 1$$

The reported value of the fund is:

$$R_t = \lambda(N_{t-1}(1 + r_t) - R_{t-1}) + R_{t-1}$$

which can be rearranged to provide the observed return r_t^o

$$\begin{aligned} r_t^o &= \frac{R_t}{R_{t-1}} - 1 \\ &= \frac{\lambda(N_{t-1}(1 + r_t) - R_{t-1})}{R_{t-1}} \\ &= \lambda \left(\frac{N_{t-1}}{R_{t-1}} \right) (1 + r_t) - \lambda \end{aligned}$$

Note that the observed first-order serial correlation is given by:

$$\rho_{1,t}^o = \frac{\text{Cov}(r_t^o, r_{t-1}^o)}{\text{Var}(r_t^o)} = \frac{\text{E}[(r_t^o - \text{E}[r_t^o])(r_{t-1}^o - \text{E}[r_{t-1}^o])]}{\text{E}[(r_t^o - \text{E}[r_t^o])^2]} \quad (5)$$

We note that the expectations of the observed return series at times t and $t - 1$ are given by:

$$\text{E}[r_t^o] = \lambda \left(\frac{N_{t-1}}{R_{t-1}} \right) (1 + m) - \lambda$$

$$\text{E}[r_{t-1}^o] = \lambda \left(\frac{N_{t-2}}{R_{t-2}} \right) (1 + m) - \lambda$$

Where m is given by:

$$m = \mu - \frac{\sigma^2}{2}$$

Working with the denominator of Equation (5):

$$\begin{aligned} &\text{E}[(r_t^o - \text{E}[r_t^o])^2] \\ &\text{E} \left[\left(\left[\lambda \left(\frac{N_{t-1}}{R_{t-1}} \right) (1 + r_t) - \lambda \right] - \left[\lambda \left(\frac{N_{t-1}}{R_{t-1}} \right) (1 + m) - \lambda \right] \right)^2 \right] \\ &\lambda^2 \left(\frac{N_{t-1}}{R_{t-1}} \right)^2 \text{E} [((1 + r_t) - (1 + m))^2] \\ &\lambda^2 \left(\frac{N_{t-1}}{R_{t-1}} \right)^2 \text{E} [(1 + r_t)^2 - 2(1 + r_t)(1 + m) + (1 + m)^2] \end{aligned}$$

$$\begin{aligned}
& \lambda^2 \left(\frac{N_{t-1}}{R_{t-1}} \right)^2 \left(\mathbb{E} [(1+r_t)^2] - 2(1+m)(1+m) + (1+m)^2 \right) \\
& \lambda^2 \left(\frac{N_{t-1}}{R_{t-1}} \right)^2 \left(\mathbb{E} [1+2r_t+r_t^2] - (1+m)^2 \right) \\
& \lambda^2 \left(\frac{N_{t-1}}{R_{t-1}} \right)^2 (1+2m+\sigma^2+m^2 - (1+m)^2) \\
& \lambda^2 \left(\frac{N_{t-1}}{R_{t-1}} \right)^2 ((1+m)^2 + \sigma^2 - (1+m)^2) \\
& \lambda^2 \left(\frac{N_{t-1}}{R_{t-1}} \right)^2 \sigma^2
\end{aligned}$$

And now the numerator of Equation (5):

$$\begin{aligned}
& \mathbb{E} \left[(r_t^o - \mathbb{E}[r_t^o]) (r_{t-1}^o - \mathbb{E}[r_{t-1}^o]) \right] \\
& \mathbb{E} \left[r_t^o r_{t-1}^o - \mathbb{E}[r_t^o] r_{t-1}^o - \mathbb{E}[r_{t-1}^o] r_t^o + \mathbb{E}[r_t^o] \mathbb{E}[r_{t-1}^o] \right] \\
& \mathbb{E} [r_t^o r_{t-1}^o] - \mathbb{E}[r_t^o] \mathbb{E}[r_{t-1}^o] - \mathbb{E}[r_{t-1}^o] \mathbb{E}[r_t^o] + \mathbb{E}[r_t^o] \mathbb{E}[r_{t-1}^o] \\
& \mathbb{E} [r_t^o r_{t-1}^o] - \mathbb{E}[r_t^o] \mathbb{E}[r_{t-1}^o] \\
& \mathbb{E} \left[\left(\lambda \left(\frac{N_{t-1}}{R_{t-1}} \right) (1+r_t) - \lambda \right) \left(\lambda \left(\frac{N_{t-2}}{R_{t-2}} \right) (1+r_{t-1}) - \lambda \right) \right] - \mathbb{E}[r_t^o] \mathbb{E}[r_{t-1}^o] \\
& \mathbb{E} \left[\lambda \left(\frac{N_{t-1}}{R_{t-1}} \right) \lambda \left(\frac{N_{t-2}}{R_{t-2}} \right) (1+r_t)(1+r_{t-1}) \right] \\
& \quad - \lambda \mathbb{E}[r_t^o] - \lambda^2 - \lambda \mathbb{E}[r_{t-1}^o] - \lambda^2 + \lambda^2 - \mathbb{E}[r_t^o] \mathbb{E}[r_{t-1}^o] \\
& \mathbb{E} \left[\lambda \left(\frac{N_{t-1}}{R_{t-1}} \right) \lambda \left(\frac{N_{t-2}}{R_{t-2}} \right) (1+r_t)(1+r_{t-1}) \right] \\
& \quad - \left(\lambda \mathbb{E}[r_t^o] + \lambda \mathbb{E}[r_{t-1}^o] + \lambda^2 + \mathbb{E}[r_t^o] \mathbb{E}[r_{t-1}^o] \right) \\
& \mathbb{E} \left[\lambda \left(\frac{N_{t-1}}{R_{t-1}} \right) \lambda \left(\frac{N_{t-2}}{R_{t-2}} \right) (1+r_t)(1+r_{t-1}) \right] \\
& \quad - \left((\mathbb{E}[r_t^o] + \lambda) (\mathbb{E}[r_{t-1}^o] + \lambda) \right) \\
& \mathbb{E} \left[\lambda \left(\frac{N_{t-1}}{R_{t-1}} \right) \lambda \left(\frac{N_{t-2}}{R_{t-2}} \right) (1+r_t)(1+r_{t-1}) \right] \\
& \quad - \left(\left(\lambda \left(\frac{N_{t-1}}{R_{t-1}} \right) (1+m) - \lambda + \lambda \right) \left(\lambda \left(\frac{N_{t-2}}{R_{t-2}} \right) (1+m) - \lambda + \lambda \right) \right) \\
& \mathbb{E} \left[\lambda \left(\frac{N_{t-1}}{R_{t-1}} \right) \lambda \left(\frac{N_{t-2}}{R_{t-2}} \right) (1+r_t)(1+r_{t-1}) \right] \\
& \quad - \left(\lambda \left(\frac{N_{t-1}}{R_{t-1}} \right) \lambda \left(\frac{N_{t-2}}{R_{t-2}} \right) (1+m)^2 \right) \\
& \lambda \left(\frac{N_{t-1}}{R_{t-1}} \right) \lambda \left(\frac{N_{t-2}}{R_{t-2}} \right) \mathbb{E}[1+r_t+r_{t-1}+r_t r_{t-1}]
\end{aligned}$$

$$\begin{aligned}
& - \left(\lambda \left(\frac{N_{t-1}}{R_{t-1}} \right) \lambda \left(\frac{N_{t-2}}{R_{t-2}} \right) (1+m)^2 \right) \\
\lambda \left(\frac{N_{t-1}}{R_{t-1}} \right) \lambda \left(\frac{N_{t-2}}{R_{t-2}} \right) & \left(1+m+m+E[r_t r_{t-1}] \right) \\
& - \left(\lambda \left(\frac{N_{t-1}}{R_{t-1}} \right) \lambda \left(\frac{N_{t-2}}{R_{t-2}} \right) (1+m)^2 \right) \\
\lambda \left(\frac{N_{t-1}}{R_{t-1}} \right) \lambda \left(\frac{N_{t-2}}{R_{t-2}} \right) & \left(1+2m+m^2-m^2+E[r_t r_{t-1}] \right) \\
& - \left(\lambda \left(\frac{N_{t-1}}{R_{t-1}} \right) \lambda \left(\frac{N_{t-2}}{R_{t-2}} \right) (1+m)^2 \right) \\
\lambda \left(\frac{N_{t-1}}{R_{t-1}} \right) \lambda \left(\frac{N_{t-2}}{R_{t-2}} \right) & \left((1+m)^2+E[r_t r_{t-1}]-m^2 \right) \\
& - \left(\lambda \left(\frac{N_{t-1}}{R_{t-1}} \right) \lambda \left(\frac{N_{t-2}}{R_{t-2}} \right) (1+m)^2 \right) \\
\lambda \left(\frac{N_{t-1}}{R_{t-1}} \right) \lambda \left(\frac{N_{t-2}}{R_{t-2}} \right) & \left(E[r_t r_{t-1}]-m^2 \right)
\end{aligned}$$

We can now write the observed first-order auto correlation as

$$\begin{aligned}
\rho_{1,t}^o &= \frac{\lambda \left(\frac{N_{t-1}}{R_{t-1}} \right) \lambda \left(\frac{N_{t-2}}{R_{t-2}} \right) \left(E[r_t r_{t-1}] - m^2 \right)}{\lambda^2 \left(\frac{N_{t-1}}{R_{t-1}} \right)^2 \sigma^2} \\
&= \frac{\left(\frac{N_{t-2}}{R_{t-2}} \right) \left(E[r_t r_{t-1}] - m^2 \right)}{\left(\frac{N_{t-1}}{R_{t-1}} \right) \sigma^2} \\
&= \frac{\left(\frac{N_{t-2}}{R_{t-2}} \right) \left(\frac{E[r_t r_{t-1}] - m^2}{\sigma^2} \right)}{\left(\frac{N_{t-1}}{R_{t-1}} \right)} \\
&= \frac{\left(\frac{N_{t-2}}{R_{t-2}} \right)}{\left(\frac{N_{t-1}}{R_{t-1}} \right)} \rho_{1,t} \\
&= \left(\frac{N_{t-2}}{N_{t-1}} \right) \left(\frac{R_{t-1}}{R_{t-2}} \right) \rho_{1,t} \\
&= \left(\frac{1+r_{t-1}^o}{1+r_{t-1}} \right) \rho_{1,t} \\
&= \left(\frac{\rho_{1,t}}{1+r_{t-1}} \right) \left[1 + \lambda \left(\frac{N_{t-2}}{R_{t-2}} \right) (1+r_{t-1}) - \lambda \right] \\
&= \left(\frac{\rho_{1,t}}{1+r_{t-1}} \right) (1-\lambda) + \lambda \left(\frac{N_{t-2}}{R_{t-2}} \right) \rho_{1,t}
\end{aligned}$$

where $\rho_{1,t}$ is the actual first-order serial correlation in the underlying series r_t . Note that:

$$N_t = N_0 \prod_{\tau=1}^t (1+r_\tau)$$

and

$$\begin{aligned}
R_t &= \lambda(N_{t-1}(1+r_t) - R_{t-1}) + R_{t-1} \\
&= \lambda N_t + R_{t-1}(1-\lambda) \\
&= \lambda(1-\lambda)^0 N_t + \lambda(1-\lambda)N_{t-1} + \lambda(1-\lambda)^2 N_{t-2} + (1-\lambda)^3 R_{t-3} \\
&= \lambda \left[\sum_{n=0}^t (1-\lambda)^n N_{t-n} \right] \\
&= \lambda \left[\sum_{n=0}^t (1-\lambda)^n N_0 \prod_{\tau=1}^{t-n} (1+r_\tau) \right]
\end{aligned}$$

Going back to our expression for $\rho_{1,t}^o$,

$$\begin{aligned}
\rho_{1,t}^o &= \left(\frac{\rho_{1,t}}{1+r_{t-1}} \right) (1-\lambda) + \lambda \left(\frac{N_{t-2}}{R_{t-2}} \right) \rho_{1,t} \\
&= \left(\frac{\rho_{1,t}}{1+r_{t-1}} \right) (1-\lambda) + \left(\frac{N_0 \prod_{\tau=1}^{t-2} (1+r_\tau)}{\sum_{n=0}^{t-2} (1-\lambda)^n N_0 \prod_{\tau=1}^{t-n-2} (1+r_\tau)} \right) \rho_{1,t} \\
&= \left(\frac{\rho_{1,t}}{1+r_{t-1}} \right) (1-\lambda) + \left(\frac{\prod_{\tau=1}^{t-2} (1+r_\tau)}{\sum_{n=0}^{t-2} (1-\lambda)^n \prod_{\tau=1}^{t-n-2} (1+r_\tau)} \right) \rho_{1,t} \\
&= a(1-\lambda) + b
\end{aligned}$$

Note that both a and b are less than 1.0.

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