Operational Risk

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Abstract

This paper studies operational risk. We discuss its economic and mathematical characterization and its estimation. The insights for this characterization originate in the corporate finance and credit risk literature. Operational risk is of two types, either (i) the risk of a loss due to the firm's operating technology, or (ii) the risk of a loss due to agency costs. These two types of operational risks generate loss processes with completely different characteristics. The mathematical characterization of these operational risks is modeled after the risk of default in the reduced form credit risk literature. We show that although it is conceptually possible to estimate the operational risk processes' parameters using only market prices, the non-observability of the firm's value makes this an unlikely possibility, except in rare cases. Instead, we argue that data internal to the firm, in conjunction with standard hazard rate estimation procedures, provides a more fruitful alternative. Finally, we show that the inclusion of operational risk into the computation of fair economic capital (as with revised Basel II) without the consideration of a firm's NPV, will provide biased (too large) capital requirements.

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1 Introduction

Risk management concerns the investigation of four significant risks of a loss to a firm or portfolio: market risk, credit risk, liquidity risk, and operational risk (see Jarrow and Turnbull [13], p. 587). Market risk includes the risk of a loss due to unanticipated price movements in financial securities or asset values, and it includes price fluctuations due to either equities, interest rates, commodities, or foreign currencies. Credit risk is the risk of a loss due to default, and liquidity risk is the risk of a loss due to the inability to liquidate an asset or financial position at a reasonable price in a reasonable time period. And, according to the revised Basel Committee revised report [1] "operational risk is defined as the risk of loss resulting from the inadequate or failed internal processes, people and systems or from external events. This definition includes legal risk." Furthermore, "legal risk includes, but is not limited to, exposure to fines, penalties, or punitive damages resulting from supervisory actions, as well as private settlements."

A study of the academic risk management/financial engineering literature readily confirms that the field has mastered - at least conceptually - market and credit risk (for texts on these topics, see Jarrow and Turnbull [13], Bielecki and Rutkowski [3], and Musiela and Rutkowski [18]). Most recently, liquidity risk has become the focus of a significant research effort (see Jarrow and Protter [14] for a review). In contrast, operational risk has received very little academic scrutiny (see Jorion [16], chapter 19, Leippold and Vanini [17] and the few references cited therein[1]). The available literature on operational risk is almost exclusively contained in the institutional press.[2] The purpose of this paper is to fill this void in the academic literature by providing an economic and mathematical characterization of operational risk, useful for quantification and estimation. This characterization is based on insights from the corporate finance and credit risk literatures. Estimation of the model's parameters is left for subsequent research.

Operational risk is partitioned into one of two types, either (i) the risk of a loss due to the firm's operating technology/system, including failed internal processes and transactions, or (ii) the risk of a loss due to agency costs, including fraud and mismanagement. These two types of operational risks generate loss processes with completely different characteristics. One is based on the process/system, the other is based on incentives. The mathematical characterization for both of these operational risks is modeled similar to the modeling of default risk in the reduced form credit risk literature. We show that although it is conceptually possible to estimate the operational risk processes' parameters

[1] Leippold and Vanini [17] study operational risk from a production (network) perspective. Leippold and Vanini's approach differs from that taken in our paper.

[2] See for example, "Op Risk and Black Swans," *Risk magazine*, Sept. 2004.

using only market prices, the non-observability of the firm's value makes this an unlikely possibility, except in rare cases. Instead, we argue that data internal to the firm, in conjunction with standard hazard rate estimation procedures, provides a more fruitful alternative. Its estimation requires time series observations of gains and losses generated by the firm's own operating system. Only the agency cost component can possibly be estimated using data external to the firm's operating system.

One additional minor, but important contribution of this paper, is our highlighting the importance of both the identification and inclusion of the firm's NPV process in the computation of risk measures (like Value at Risk) for the determination of a firm's fair economic capital. Unlike market, credit and liquidity risk which involve markets that are *external* to the firm, both operational risk and a firm's NPV are *internal* to the firm. The inclusion of operational risk in capital determination, without the NPV process, ignores a fundamental economic motivation for the existence of firms, and this procedure will provide biased (too large) estimates of a firm's fair economic capital (as is the case in the revised Basel II report [1]).

An outline of this paper is as follows. Section 2 provides the economic based definition. Section 3 provides a simple, yet robust mathematical characterization of the operational event risk processes. Section 4 discusses estimation, and section 5 concludes.

2 The Definition

This section provides an economic and mathematical characterization of operational risk. Based on the standard definition, we divide operational risk into two types. Type one corresponds to the risk of a loss due to the firm's operating *system*, i.e. a failure in a transaction or investment, either due to an error in the back office (or production) process or due to legal considerations. And, type two corresponds to the risk of a loss due to *incentives*, including both fraud and mismanagement.[3] The second type of operational risk represents an agency cost, due to the separation of a firm's ownership and management. Agency costs are recognized as a significant force in economics, and they have received significant study in the corporate finance literature as key determinants of the firm's capital structure and dividend policy (see Brealey and Myers [4]). Both types of operational risk losses occur with repeated regularity, and they can be small or catastrophic. Spectacular catastrophic examples include Orange County, the Barings Bank failure, or the Bankers Trust and Procter and Gamble fiasco (see Risk Books [21]).

We consider a finite horizon, continuous trading setting with $t \in [0, T]$ on a filtered probability space $(\Omega, F, \{F_t\}, P)$ satisfying the usual conditions (see Protter [19]) with P the statistical probability measure. Let r_t denote the

[3] There is some ambiguity with respect to the classification of human error. If the human error is due to misaligned incentives, then it should be included in the agency operational risk category. Otherwise, it is system related risk.

default free spot rate of interest, and let S_t^i represent the market value of asset $i \in \{1, ..., n\}$ at time t. For simplicity, we assume that these assets have no cash flows over their lives.[4] We assume that the market for these assets is arbitrage free, so that there exists an equivalent martingale probability measure Q such that

$$S_t^i = E_t^Q\{S_T^i e^{-\int_t^T r_v dv}\} \text{ for all } i$$

where $E_t^Q\{\cdot\}$ is under the martingale probability. Under this structure, market prices can be computed as the expected discounted value of their future cash flows under the martingale probability. The markets need not be complete, so the martingale probability need not be unique.

We consider a firm operating in this setting, trading financial securities or investing in real assets with prices S_t^i for $i = 1, ..., n$, generating a firm value of V_t. As indicated, the firm is conceptualized as a portfolio of financial securities and/or real assets. The firm's value represents the aggregate value of the left side of the firm's balance sheet. The right side of the firm's balance sheet consists of the firm's liabilities and equity, which we also assume trade in an arbitrage free, but possibly incomplete market. Under this assumption, the firm's value V_t trades. Consequently, its time t value can also be represented as an expected discounted future value using the martingale measure, i.e.

$$V_t = E_t^Q\{V_T e^{-\int_t^T r_v dv}\}.$$

The economic setting can be understood by examining Figure 1. The firm is represented by an operating technology (the green box) that takes as inputs traded financial securities and real assets, with prices S_t^i, and returns as an output the value of the firm V_t. The operating technology transformation represented within the firm is discussed subsequently. Also, for subsequent usage, we let $\mathbf{X}_t$ denote a vector of state variables, $F_t - measurable$, that characterize the state of the economy at time t. Included in this set of state variables are the spot rate of interest r_t and the market prices S_t^i. We let F_t^X denote the (completed to include all zero probability events) filtration generated by the state variables $\mathbf{X}_t$ up to and including time t.

2.1 The NPV of the Firm's Operating Technology

In our setting, the existence of an operating technology distinguishes a firm from an individual trading in the market. An operating technology transforms the assets the firm purchases into more valuable objects. As argued in Jarrow and Purnanandam [15], this is possible if the firm has some special talent, information, or managerial expertise in selecting assets for investment (see the discussion section below). This increase in asset value due to the firm's operating technology is represented by the $F_t - measurable$ stochastic process $\pi(t) \geq 1$ for all t. Letting S_t represent the aggregate market value of the firm's asset

[4] Cash flows only complicate the notation, but not the logic of the subsequent analysis, and are therefore omitted for clarity.

portfolio, the change in value due to the firm's operating technology at time t is

$$S_t \cdot \pi(t) \geq S_t,$$

see Figure 1. $\pi(t)$ is called the firm's net present value (NPV) process. The firm's NPV process will be shown to be an important determinant for the understanding and quantification of a firm's operational risk.

2.2 An Owner-Managed Firm

To introduce the characterization of operational risk, we start with the simplest economic setting - that of an owner-managed firm (or portfolio). The owner or manager invests his capital in some assets, either real or financial and she is concerned about the risk of a loss from her investments.

As in the previous section, the time t marked-to-market value of the firm's asset portfolio is denoted S_t. This value represents the market value that would be recorded, for example, if the assets were financial securities trading on an organized exchange. In a traditional risk management model, one would be concerned only with the probability distribution of this price process S_t, and various risk measures could be used to characterize the risk of a loss on this position. The loss could be due to either market, credit or liquidity risk. Operational risk, however, is something more. By construction, an owner-managed firm only faces the first type of operational risk. To an owner-managed firm, it would represent the risk due to a failed transaction, perhaps an error in the executive of a trade (or investment), a legal dispute, or possibly an error in judgement.

We represent this additional loss by the multiplication of the previous portfolio's value $S_t \cdot \pi(t)$ by the $F_t - measurable$ stochastic process $\theta_1(t) \leq 1$ for all t. The quantity $\theta_1(t)$ represents the accumulated time t "recovery" value after all operational risk events of the first type are incurred. The internal value of the firm's assets, after the inclusion of operational risk, is thus

$$S_t \cdot \pi(t) \cdot \theta_1(t),$$

see Figure 1. It is important to emphasize that this reduction in asset value due to type 1 operational risk is realized internally within the firm. Given that V_t is a traded asset, the market value of the owner-managed firm's assets is

$$V_t = E_t^Q\{S_T \cdot \pi(T) \cdot \theta_1(T) \cdot e^{-\int_t^T r_v dv}\}. \tag{1}$$

2.3 An Agent-Managed Firm

Next, we consider a firm managed by an agent. Of course, for practical applications, this is the most relevant case. Given an agent-managed firm, the second type of operational risk is also present. We represent this additional loss by the multiplication of the previous firm value $S_t \cdot \pi(t) \cdot \theta_1(t)$ by the $F_t - measurable$ stochastic process $\theta_2(t) \leq 1$ for all t which represents the accumulated time t

"recovery" value after all operational risk events of the second type. Since an agent-managed firm also faces the first type of operational risk, the value of the firm's investment portfolio must reflect both risks.

By the same line of reasoning as that given before, the internal valuation of the firm's portfolio is

$$S_t \cdot \pi(t) \cdot \theta_1(t) \cdot \theta_2(t),$$

see Figure 1, and the time t market value of the firm's assets is

$$V_t = E_t^Q\{S_T \cdot \pi(T) \cdot \theta_1(T) \cdot \theta_2(T) \cdot e^{-\int_t^T r_v dv}\}. \tag{2}$$

This is the most general formulation of operational risk that we will consider below.

2.4 The Firm's NPV Process - Revisited

It is important to stress the economic importance of including the firm's NPV process when considering operational risk for the purposes of risk measure computations, as with the revised Basel II capital requirements [1]. As verified by expression (2), operational risk reduces the firm's value, relative to market prices (marking-to-market), because operational risk always has a non-positive impact on firm value ($\theta_1(T) \cdot \theta_2(T) \leq 1$). This implies that, without the inclusion of the nonnegative NPV process ($\pi(t) \geq 1$), there would be no reason for this firm to exist. Indeed, shareholders could generate higher wealth themselves by directly purchasing the firm's assets. This statement follows from that fact that without the NPV process included, $V_t \leq S_t$ for all t with probability one.[5] In fact, in well-functioning financial markets, we would expect that for healthy firms, the reverse inequality $V_t \geq S_t$ holds for all t with probability one. That is, on average (as measured by expression (2) in market prices), the NPV process dominates (is larger than) the average loss due to operational risk. This is an important observation for the determination of a firm's fair economic capital.

Unlike market, credit or liquidity risk which involve market dynamics and are *external* to the firm, operational risk is *internal* to the firm. When considering risks that are internal to the firm, it is essential to include both the negative (operational) as well as the positive (NPV) ones. The inclusion of the negative alone - operational risk - without the consideration of the positive one - the firm's NPV process - ignores a fundamental economic element that is essential for the firm's existence - the reason why a firm's equity trades at positive values. Of course, the NPV dominates operational risk only on average, and in the tail of the distribution, operational risk may far exceed the firm's NPV (in fact, it could exceed both the firm's NPV and the market price component). As such, operational risk is important, and should be included in the determination of economic capital. But, if one wants to determine a firm's *fair* economic capital, then reducing the firm's value due to operational risk (as in revised Basel II), without the inclusion of a firm's NPV, is inappropriate.

[5] The algebra is that without $\pi(t)$, $\widehat{V}_t = E_t^Q\{S_T \cdot \theta_1(T) \cdot \theta_2(T) \cdot e^{-\int_t^T r_v dv}\} \leq E_t^Q\{S_T e^{-\int_t^T r_v dv}\} = S_t$.

2.5 Discussion

Although we believe that this characterization of the firm's NPV and operational risk processes as in Figure 1 is uniformly applicable to both financial and non-financial firms, to help the reader reach a similar conclusion, this section discusses some illustrations. In these illustrations, the NPV process is due to the firm's (or business line's) *franchise* value.

For commercial and investment banks, this characterization captures its various lines of businesses, including for example sales and trading, credit cards, and consumer deposits. For sales and trading, the NPV process is generated by proprietary trading in undervalued securities or offering services to its clients. Systems operational risk occurs due to failed transactions, and agency operational risk is due to fraud and/or mismanagement, examples include Bankers Trust and Barings Bank (Risk Books [21], chp 36 and 37). It is interesting to point out that the NPV process has long been recognized, and even estimated, for credit cards and consumer deposits (see Chatterjea, Jarrow and Neal [6] and Janosi, Jarrow and Zullo [10]). In this context, the NPV process is due to the fact that banks can pay below market rates on demand deposits or charge above market rates on credit card loans.

The same characterization applies to mutual funds and hedge funds. For mutual funds, their franchise value is enormous (the NPV process), their system operational risk is controlled by recording/confirming all transactions, and their agency operational risk is controlled by monitoring management trading on their own accounts. Even so, failures sometimes occur on both dimensions, e.g. the front running of trades for some clients at the expense of other clients, and failed market manipulations. For hedge funds, a similar logic applies for both the NPV and operational risk processes. Long Term Capital Management is the most prominent example of a hedge fund with a large NPV process based on arbitrage trading strategies, but eventually failing due to agency cost operational risks. According to some accounts, LTCM extended its trading to asset classes where it did not have expertise, and it miscalculated its liquidity risk (see Risk Books [21], chp 32).

For non-financial firms, the science of selecting investment projects - capital budgeting - is the science of computing the NPV process (see Ross, Westerfield, Jaffee [20]). System operational risks are failures in the production process, a well-studied area in operations management. Agency operational risk is manifested in various guises. For drug companies, it could be the occurrence of a seriously detrimental (or fatal) side effect for drugs resulting from a hasty entrance into a new market. For fast food companies it could be food poisoning due to sloppy food preparation. And, for construction companies, it could be faulty construction or the use of inadequate building materials.

Although not exhaustive, this varied list of illustrations documents the generality of our characterization of the firm's value process. For practical applications, we need to impose more structure on these abstract processes. This is the content of the next section.

3 The NPV and Operational Risk Processes

This section presents a simple, yet robust formulation of the NPV and operational risk event processes. We model these stochastic processes following the jump process formulation often used in the reduced form credit risk literature (see Bielecki and Rutkowski [3]). Other formulations are possible, and these extensions are left for subsequent research.

3.1 The NPV Process

This section presents the stochastic process for the NPV process $\pi(t)$. Let $N_0(t)$ be a doubly stochastic (Cox) counting process, initialized at zero ($N_0(0) = 0$), that counts the number of positive NPV events that occur between time 0 up to and including time t. We assume that this counting process is measurable with respect to the given filtration F_t, and that it has an intensity per unit time given by $\lambda_0(s) = \lambda_0(\mathbf{X}_s) \geq 0$ that is $F_s^X - measurable$. Given F_T^X, we assume that $N_0(t)$ is independent of S_t. This is often called the "conditional independence" assumption. Continuing, we let a NPV event at time t cause a percentage increase in firm value equal to $\alpha(t) = \alpha(\mathbf{X}_t) > 0$ that is also $F_t^X - measurable$.

Thus,

$$\pi(t) = \prod_{i=0}^{N_0(t)} (1 + \alpha_{T_i}). \tag{3}$$

where $\pi(0) = 1$, T_i for $i = 0, 1, 2, ...$ are the jump times of $N_0(t)$ with $T_0 = \alpha_{T_0} = 0$. We assume that the intensity and drifts processes satisfy the technical conditions needed for the existence of the various processes and the subsequent computations (see Bremaud [5]).

Note that in general, the NPV process $\pi(t)$ is correlated to the portfolio's market value S_t due to their mutual dependence on the state variables $\mathbf{X}_t$. However, after conditioning upon these variables, the randomness generating the counting process $N_0(t)$ in the asset's NPV is idiosyncratic and firm specific, not otherwise related to market prices or the state variables. Even so, this jump NPV process risk could still require a market risk premium if these risks are not diversifiable in a large portfolio (see Jarrow, Lando, Yu [12]). To accommodate this possibility, under the equivalent martingale measure Q, we let the counting process have the intensity $\lambda_0(s)\mu_0(s)$ where $\mu_0(s) > 0$ is the risk premium associated with the NPV process (see Bremaud [5] p. 241).

3.2 System Type Operational Risk

First, let us consider the first type of operational risk. We want a formulation of the $\theta_1(t)$ process that can be utilized in practice. Conceptually, it is reasonable to believe that the occurrence of an operational risk event is related to the volume of transactions underlying the firm's portfolio. Such a detailed implementation would require decomposing the firm's portfolio into its component parts, and then modeling the trading process of each individual asset, keeping track of the number of transactions and the operational risk events. The occurrence of an operational risk event could then be modeled at the transaction level. This would be a very complex procedure (see Leippold and Vanini [17] for one such approach). Although perfectly reasonable to pursue, we follow a "reduced form approach" instead and concentrate on the entire portfolio's value.[6] Refinements in this methodology are delegated to subsequent research.

Formally, let $N_1(t)$ be another doubly stochastic (Cox) counting process, initialized at zero ($N_1(0) = 0$), that counts the number of operational risk events of type 1 that occur between time 0 up to and including time t. We assume that this counting process is measurable with respect to the given filtration F_t, and that it has an intensity per unit time given by $\lambda_1(s) = \lambda_1(\mathbf{X}_s) \geq 0$ that is $F_s^X - measurable$. Given F_T^X, we assume that $N_1(s)$ is independent of S_t and $N_0(t)$. We let an operational risk event at time t cause a percentage reduction in firm value equal to $-1 < \delta_1(t) = \delta_1(\mathbf{X}_t) < 0$ that is $F_t^X - measurable$. Hence,

$$\theta_1(t) = \prod_{i=0}^{N_1(t)} (1 + \delta_1(T_i)) \tag{4}$$

where $\theta_1(0) = 1$, T_i for $i = 0, 1, 2, ...$ are the jump times of $N_1(t)$ with $T_0 = \delta_1(T_0) = 0$.

Similar to the firm's NPV process, we assume that conditional upon the path of the state variables $\mathbf{X}_t$ up to time T, $N_1(t)$ is independent of both S_t and $N_0(t)$. Again, this assumption is equivalent to saying that any additional randomness present in this operational risk is idiosyncratic and firm specific. As before, this risk could still require a market risk premium if these risks are not diversifiable in a large portfolio. To accommodate this possibility, under the equivalent martingale measure Q, we let the counting process have the intensity $\lambda_1(s)\mu_1(s)$ where $\mu_1(s) > 0$ is the risk premium associated with this operational risk process.

Also, unconditionally, all of the $N_1(t), N_0(t)$ and S_t are correlated processes. They are correlated through the common state variables. This correlation is important, for example, because one might believe that as the value of the firm's asset portfolio increases, the size of the system type operational risk losses may decline due to an improved back office systems.

3.3 Agency Cost Type Operational Risk

The agency cost operational risk event can be modeled in the same way. Using superscripts "2" to indicate a type 2 operational risk event, we have

$$\theta_2(t) = \prod_{i=0}^{N_2(t)} (1 + \delta_2(T_i)) \tag{5}$$

where $\theta_2(0) = 1$, T_i for $i = 0, 1, 2, ...$ are the jump times of $N_2(t)$, and $T_0 = \delta_2(T_0) = 0$. $N_2(t)$ is assumed to be conditionally independent of $N_0(t), N_1(t), S_t$.

[6] In fact, this should really be thought of as the portfolios related to the various business lines as specified in the revised Basel II report [1] p. 139. The subsequent analysis follows with the appropriate and straightforward aggregation of the business lines into the firm's entire portfolio.

Again, this assumption is equivalent to saying that any additional randomness present in this operational risk is idiosyncratic and firm specific. As before, this risk could still require a market risk premium if these risks are not diversifiable in a large portfolio. To accommodate this possibility, under the equivalent martingale measure Q, we let the counting process have the intensity $\lambda_2(s)\mu_2(s)$ where $\mu_2(s) > 0$ is the risk premium associated with this operational risk process.

Finally, we note again that the agency cost operational risk process is correlated with the firm's asset portfolio, the NPV process, and the system type operational risk process through the state variables $\mathbf{X}_t$. This correlation is important because one might believe that as the value of the firm's asset portfolio declines, agency cost operational risk losses may increase due to the firm's managers trying to increase their performance and save their jobs.

The key distinction between the two types of operational risk is that, most likely, $|\delta_2| >> |\delta_1|$ and $\lambda_2 << \lambda_1$, that is operational risk of type 2 results in a larger loss, but is less likely to occur.

3.4 The Firm's Internal Value Generating Process

In conjunction, the firm's time t internal value generating process is given by

$$S_t \cdot \pi(t) \cdot \theta_1(t) \cdot \theta_2(t) = S_t \prod_{i=0}^{N_0(t)}(1+\alpha_{T_i}) \prod_{j=1}^{2}[\prod_{i=0}^{N_j(t)}(1+\delta_j(T_i))] \tag{6}$$

where the counting processes $\{N_0(t), N_1(t), N_2(t)\}$ have the intensities $\{\lambda_0(t), \lambda_1(t), \lambda_2(t)\}$ under the statistical probability measure P and $\{\lambda_0(t)\mu_0(t), \lambda_1(t)\mu_1(t), \lambda_2(t)\mu_2(t)\}$ under the martingale probability measure Q. Of course, for the computation of risk measures, like Value at Risk, the statistical measure is the relevant probability, while for valuation and hedging the martingale measure is the appropriate choice.

For practical applications, although not necessary, we assume that within a business line, the NPV and operational risk processes $(\pi(t), \theta_1(t), \theta_2(t))$ do not depend on the traded assets S_t^i (or S_t) as given in Figure 1. This implies that the firm's internal value process is linear in S_t so that the internal value process applies to the individual traded assets S_t^i as well as to portfolios of traded assets S_t. This observation is used in example 1 below without further comment.

3.5 Asset Pricing and Risk Measures

For pricing and risk measure computation, we have that the market value of the firm's value generating process can be represented as

$$V_t = E_t^Q\{S_T e^{-\int_t^T r_v dv} \prod_{i=0}^{N_0(T)}(1+\alpha_{T_i}) \prod_{j=1}^{2}[\prod_{i=0}^{N_j(T)}(1+\delta_j(T_i))]\}. \tag{7}$$

We show in the appendix that this can be simplified to

$$V_t = E_t^Q\{S_T e^{-\int_t^T [r_s - \alpha_s\lambda_0(s)\mu_0(s) - \delta_1(s)\lambda_1(s)\mu_1(s) - \delta_2(s)\lambda_2(s)\mu_2(s)]ds}\}. \tag{8}$$

We see that the firm value equals the portfolio's market price process S_T discounted by the spot rate after an adjustment for the NPV process and operational risk. The spot rate process is decreased by the expected increase in value due to the NPV process (recall that α_s is positive), but increased to reflect both types of operational risk (recall that both $\delta_1(s)$ and $\delta_2(s)$ are negative). This simplification is important because it demonstrates that pricing in the presence of operational risk can be handled via a simple adjustment to the discount rate (this same adjustment is used in the credit risk literature). Then, the direct application of the mathematics developed for the pricing of interest rate derivatives under default free term structure evolutions can be directly applied to the computation of the relevant quantities given operational risk. For example, if one assumes affine processes for the combined jumps and recovery rate processes, then closed form solutions for these expressions and various options on the firm's cash flows can be obtained (see Shreve [22] chapter 10). To illustrate these computations, we provide the following two-factor Gaussian example.

Example 1 *Two-Factor Affine Model*

Consider a two factor Gaussian model as in Shreve [22], p. 406). The CIR two factor model follows similarly (see p. 420 instead). Here, the state variables follow diffusion processes given by

$$dX_1(t) = -\phi_1 X_1(t)dt + dW_1(t)$$

$$dX_2(t) = -\phi_{21} X_1(t)dt - \phi_2 X_2(t) + dW_2(t)$$

where $W_1(t)$ and $W_2(t)$ are independent Brownian motions under Q, and $\phi_1 > 0$, $\phi_2 > 0$, ϕ_{21} are constants.

Let the spot rate follow an affine process in the state variables given by

$$r_t = a_0 + a_1 X_1(t) + a_2 X_2(t).$$

The value of a default free zero-coupon bond in the market (where $S_T = 1$ with probability one) is

$$S_t = E_t^Q\{1 \cdot e^{-\int_t^T r(u)du}\} = e^{-X_1(t)C_1(T-t)-X_2(t)C_2(T-t)-A(T-t)}$$

where $C_1(0) = C_2(0) = A(0) = 0$. If $a_1 \neq a_2$,

$$C_1(\tau) = \frac{1}{\phi_1}\left(a_1 - \frac{\phi_{21}a_2}{a_1}\right)(1 - e^{-\phi_1\tau}) + \frac{\phi_{21}a_2}{a_2(a_1 - a_2)}(e^{-\phi_2\tau} - e^{-\phi_1\tau}),$$

$$C_2(\tau) = \frac{a_2}{\phi_2}(1 - e^{-\phi_2\tau}),$$

$$A(\tau) = \int_0^\tau \left(-\frac{1}{2}C_1^2(u) - \frac{1}{2}C_2^2(u) + a_0\right) du.$$

Now, let the NPV process and operational risk event intensities, under Q, also satisfy an affine process in the state variables given by

$$\begin{aligned}
\alpha_t \lambda_0(t)\mu_0(t) &= b_0 + b_1 X_1(t) + b_2 X_2(t), \\
\delta_1 \lambda_1(t)\mu_1(t) &= c_0 + c_1 X_1(t) + c_2 X_2(t), \\
\delta_2 \lambda_2(t)\mu_2(t) &= d_0 + d_1 X_1(t) + d_2 X_2(t).
\end{aligned}$$

Then, define new parameters by

$$\begin{aligned}
\psi_0 &= a_0 - b_0 - c_0 - d_0, \\
\psi_1 &= a_1 - b_1 - c_1 - d_1, \\
\psi_2 &= a_2 - b_2 - c_2 - d_2,
\end{aligned}$$

and an adjusted spot rate process by

$$\begin{aligned}
R(t) &= r_t - \alpha_t \lambda_0(t)\mu_0(t) - \delta_1 \lambda_1(t)\mu_1(t) - \delta_2 \lambda_2(t)\mu_2(t) \\
&= \psi_0 + \psi_1 X_1(t) + \psi_2 X_2(t).
\end{aligned}$$

The value within the firm for a traded Treasury zero-coupon bond, according to expression (8), is

$$V_t = E_t^Q\{1 \cdot e^{-\int_t^T R(u)du}\} = e^{-X_1(t)\widetilde{C}_1(T-t) - X_2(t)\widetilde{C}_2(T-t) - \widetilde{A}(T-t)}$$

where $\widetilde{C}_1(0) = \widetilde{C}_2(0) = \widetilde{A}(0) = 0$. *If* $\psi_1 \neq \psi_2$, *then*

$$\widetilde{C}_1(\tau) = \frac{1}{\phi_1}\left(\psi_1 - \frac{\phi_{21}\psi_2}{\psi_1}\right)(1 - e^{-\phi_1 \tau}) + \frac{\phi_{21}\psi_2}{\psi_2(\psi_1 - \psi_2)}(e^{-\phi_2 \tau} - e^{-\phi_1 \tau}),$$

$$\widetilde{C}_2(\tau) = \frac{\psi_2}{\phi_2}(1 - e^{-\phi_2 \tau}),$$

$$\widetilde{A}(\tau) = \int_0^\tau \left(-\frac{1}{2}\widetilde{C}_1^2(u) - \frac{1}{2}\widetilde{C}_2^2(u) + \psi_0\right) du.$$

These two values for the default free zero-coupon bond differ by the NPV and operational risk processes impact within the firm. In general, we would expect that $V_t \geq S_t$.

Expression (6) is also directly relevant for computing various risk management measures. For example, computing the 5% Value at Risk measure over the horizon $[0, T]$ for the firm's asset value requires finding the smallest $\eta > 0$ such that

$$P(V_T \leq -\eta) = .05.$$

Using expression (6) yields

$$P(S_T \prod_{i=0}^{N_0(T)}(1 + \alpha_{T_i}) \prod_{j=1}^{2}[\prod_{i=0}^{N_j(T)}(1 + \delta_j(T_i))] \leq -\eta) = .05.$$

Given diffusion processes for the state variables $\mathbf{X}_t$, a process for S_T, e.g. geometric Brownian motion, and using the fact that $N_j(t)$ are all mutually independent given F_T^X, this is easily computed using standard Monte-Carlo techniques (see Glasserman [9]). Of course, the computation of Value at Risk is under the statistical probability measure P using the intensities $\{\lambda_0(t),\ \lambda_1(t),\ \lambda_2(t)\}$ for the relevant counting processes.

To further develop our understanding of expression (8) and its uses, it is instructive to consider the constant parameter case.

Example 2 *Constant Parameters*
Assuming that

$$\alpha_t \lambda_0(t)\mu_0(t),\ \ \delta_1(t)\lambda_1(t)\mu_1(t),\ \ \delta_2(t)\lambda_2(t)\mu_2(t)$$

are constants, expression (8) simplifies to

$$V_t = S_t(1+\alpha)^{N_0(t)}(1+\delta_1)^{N_1(t)}(1+\delta_2)^{N_2(t)} e^{[\alpha\lambda_0\mu_0+\delta_1\lambda_1\mu_1+\delta_2\lambda_2\mu_2](T-t)}. \tag{9}$$

Here, the firm's value is seen to be equal to the portfolio's market value at time t adjusted to reflect all past NPV and operational risk shocks, plus anticipated changes in these events.

In this constant parameter case, the adjustments to the marked-to market value of the firm's portfolio S_t to reflect operational risk are easy to compute. They amount to a deterministic and proportional change in value as represented by the terms following S_t in expression (9).

As a first pass in implementing operational risk into a firm's risk management procedure, the constant parameter case, expression (9), could prove a very useful tool. Its implementation would require minimal changes to any existing risk management procedure. For example, computing prices and hedges at time 0 amounts to using the following expression

$$V_0 = S_0 e^{[\alpha\lambda_0\mu_0+\delta_1\lambda_1\mu_1+\delta_2\lambda_2\mu_2]T}. \tag{10}$$

In this expression, the modification is to multiply the market value S_0 by a deterministic proportionality constant which is greater than or equal to 1 under the reasonable assumption that $V_0 \geq S_0$. For computing risk measures, like Value at Risk, one only needs to modify the existing procedure for computing the market value of the portfolio S_t by a proportionality factor, obtained by running three independent Poisson processes $\{N_0(t), N_1(t), N_2(t)\}$. In contrast, expression (8) requires the specification of stochastic processes for the same quantities and a more complex adjustment to the computation of the expectation operator (the integral) as illustrated in example 1 above.

4 Estimation

This section discusses the estimation of the NPV and operational risk processes' parameters using market prices. It is argued below that it is conceptually possible to estimate the NPV and operational risk factor parameters using only market prices. However, from a practical perspective, except in rare cases, this conceptual possibility can not be achieved. In contrast, the NPV and operational risk processes can more easily be estimated using data internal to the firm, using standard hazard rate estimation procedures. The estimation of the various risk premium relevant to operational risk may be estimated using techniques recently employed in the credit risk literature (see Driessen [7] and Berndt, et. al. [2]). The estimation of these operational risk premium will not be discussed further in this paper.

To see the validity of our assertions regarding the estimation of the NPV and operational risk processes' parameters, let us set up the preliminaries of the argument. First, the market is assumed to observe the prices of the traded assets S_t and V_t. These prices would be recorded in the financial press. We note, for subsequent usage, that V_t represents the total value of the firm's liabilities and equity. Second, the technology's NPV and operational risk factors $\pi(t), \theta_1(t), \theta_2(t)$, being firm specific and internal to the firm are not directly observable to the market. Consequently, the issue is whether one can infer the NPV and operational risk factors via market prices alone. We next argue that this is conceptually possible.

To understand why, consider expression (9), where market prices give us the left side of the following expression,

$$\frac{V_t}{S_t} = (1+\alpha)^{N_0(t)}(1+\delta_1)^{N_1(t)}(1+\delta_2)^{N_2(t)}e^{[\alpha\lambda_0\mu_0+\delta_1\lambda_1\mu_1+\delta_2\lambda_2\mu_2](T-t)}.$$

After normalizing the firm value by the market value of the underlying asset portfolio ($\frac{V_t}{S_t}$), changes in the left side represent changes in market prices due to the NPV and operational risk processes alone. When $\frac{V_t}{S_t}$ jumps, it is due to one of the counting processes $\{N_0(t),\ N_1(t),\ N_2(t)\}$[7] changing, and the percentage change in $\frac{V_t}{S_t}$ is due to the amplitude of the relevant jump process: $\{\alpha,\ \delta_1,\ \delta_2\}$. Given a reasonable collection of time series observations of the left side, it should be possible using standard statistical procedures (e.g. maximum likelihood estimation) to estimate the NPV and operational risk processes' parameters.

However, there is practical problem. As noted in the empirical literature estimating the structural approach to credit risk, the firm value process V_t is, except in rare cases, not observable. This is due to the fact that not all of the firm's liabilities and equity trade in liquid markets (e.g. unfunded pension obligations, private bank loans, lines of credit, etc.). Consequently, although conceptually possible, in most cases, market prices alone are not sufficient to estimate the NPV and operational risk processes.

An alternative and perhaps more fruitful approach for estimating the NPV and operational risk processes' parameters is to use data on these processes that are available internally to the firm. The necessary data are time series

[7]Note that two or more counting processes jumping at the same time occurs with probability zero under our structure.

observations of the dates of the occurrence of the NPV and operational risk events, and the gains/losses that result at each occurrence. Standard statistics can then be used to obtain the estimated gain/loss rates, and standard hazard rate estimation procedures can be used to obtain the intensity processes (a good source is Fleming and Harrington [8]). For example, consider a portfolio of debt securities managed by the firm. Suppose that the trading gains are regularly observed and the NPV is computed to be 5 percent with on average 20 occurrences per year. Further, the firm experiences on average 5 events of operational risk of type 1, with average losses equal to .01 percent of the portfolio's value. But, for operational risk of type 2, it only experiences 1 such event every 4 years on the portfolio, with an average loss equal to 2 percent of the portfolio's value. Then, $\widehat{\alpha} = .05$, $\widehat{\lambda}_0 = 20$, $\widehat{\delta}_1 = -.0001$, $\widehat{\lambda}_1 = 5$, and $\widehat{\delta}_2 = -.02$, $\widehat{\lambda}_2 = \frac{1}{4}$.

For agency cost type operational risk events, if large enough to be publicly reported, the counting process $N_1(t)$ is observed externally to the firm. Then, observing $\frac{V_t}{S_t}$ enables one to estimate the dollar losses ($\delta_2(s)$) directly without data internal to the firm. Indeed, one can condition on the time series observations of agency cost operational risk events and apply standard hazard rate estimation techniques (see Jarrow and Chava [11]) to estimate $\lambda_2(s)$. Given the occurrence of a catastrophic event, one can then measure the dollar losses $\delta_2(s)$ using the changes in $\frac{V_t}{S_t}$. We remark that this approach still has some remaining difficulties: (i) this approach does not include estimates for the NPV and system type operational risk parameters (α, λ_0, δ_1, λ_1), (ii) nor does it include agency cost operational loses not significant enough to be reported in the financial press, and (iii) finally, this procedure still requires an estimate of the change in the firm's value when the agency cost event occurs. These remaining difficulties are easily overcome using data internal to the firm.

5 Conclusion

This paper provides an economic and mathematical characterization of operational risk. This characterization originates in the corporate finance and credit risk literature. Operational risk is of two types, either (i) the risk of a loss due to the firm's operating technology, or (ii) the risk of a loss due to agency costs. These two types of operational risks generate loss processes with completely different characteristic, both modeled as Cox counting processes. We show that although it is conceptually possible to estimate the operational risk processes' parameters using only market prices, the non-observability of the firm's value makes this an unlikely possibility, except in rare cases. Instead, we argue that data internal to the firm, in conjunction with standard hazard rate estimation procedures, provides a more fruitful alternative. Finally, we show that the inclusion of operational risk into the computation of fair economic capital (as with revised Basel II) without the consideration of a firm's NPV, will provide biased (too large) capital requirements.

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6 Appendix

This appendix provides the proof of expression (8). First,

$$V_t = E_t^Q\{S_T e^{-\int_t^T r_v dv} \prod_{i=0}^{N_0(T)} (1+\alpha_{T_i}) \prod_{j=1}^{2} [\prod_{i=0}^{N_j(T)} (1+\delta_j(T_i))]\}. \tag{11}$$

can be rewritten as:

$$\begin{aligned} V_t &= E_t^Q\{S_T e^{-\int_t^T r_v dv} \pi(t) e^{\int_t^T \log(1+\alpha_s) dN_0(s)} \cdot \\ &\quad \theta_1(t) e^{\int_t^T \log(1+\delta_1(s)) dN_0(s)} \theta_2(t) e^{\int_t^T \log(1+\delta_2(s)) dN_2(s)}\}. \end{aligned} \tag{12}$$

Next, we note that $E_t^Q\{e^{\int_t^T \log(1+\alpha_s) dN_0(s)} \mid F_T^X\} = e^{\int_t^T \alpha_s \lambda_0(s) \mu_0(s) ds}$ with similar expressions for the two types of operational risk. Then, taking iterated conditional expectations of (12) and using conditional independence gives

$$\begin{aligned} V_t &= E_t^Q[S_T e^{-\int_t^T r_v dv} \pi(t) E_t^Q\{e^{\int_t^T \log(1+\alpha_s) dN_0(s)} \mid F_T^X\} \cdot \\ &\quad \theta_1(t) E_t^Q\{e^{\int_t^T \log(1+\delta_1(s)) dN_0(s)} \mid F_T^X\} \theta_2(t) E_t^Q\{e^{\int_t^T \log(1+\delta_2(s)) dN_2(s)} \mid F_T^X\}]. \end{aligned} \tag{13}$$

Using the note proves expression (8).

Figure 1: The Economic Setting.

S_t^i are prices of traded assets, S_t represents the aggregate value of the asset portfolio purchased by the firm, V_t is the firm's value, $\pi(t)$ is the proportionate change in the value of the firm's asset portfolio due to the firm's operating technology, $\theta_1(t)$ is the proportionate change in the value of the firm's asset portfolio due to the system operational risk, and $\theta_2(t)$ is the proportionate change in the value of the firm's asset portfolio due to the agency cost operational risk.