Recovering Heterogeneous Beliefs and Preferences from Asset Prices *

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Abstract

We propose a novel information-theoretic approach to separately identify the risk preferences and beliefs of different types of financial market investors. Investors who allocate most of their wealth in large market capitalization stocks are risk averse and believe that the aggregate stock market return is strongly countercyclical. In contrast, investors in small-growth stocks are substantially less risk averse and believe in procyclical expected stock market returns. Our findings can reconcile the procyclical expected market returns found in investor survey data with the countercyclical expected returns implied by rational expectations models.

Keywords: Conditional Euler Equations, Heterogeneous Beliefs, Heterogeneous Preferences, Relative Entropy Minimization, Smoothed Empirical Likelihood.

JEL Classification Codes: C51, E3, E70, G12, G14, G40

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I Introduction

Asset prices reflect investors' beliefs about the probability distribution of future states of the world. Therefore, understanding how investors form their beliefs and how the beliefs evolve with changing economic conditions is crucial to understanding and predicting the behavior of asset prices. This central role of beliefs explains the substantial research effort devoted to their study. Different paradigms have been proposed in the literature, ranging from rational expectations (see, e.g., Muth (1961)), which postulates that investors use all available data rationally to form their expectations about the future, to behavioral models (e.g., Barberis and Thaler (2001)), where investors are assumed to have certain biases that distort their beliefs. Unfortunately, no direct estimates of investors' beliefs are available. Researchers have largely relied on survey data, in which a group of professionals is asked to state their beliefs about various economic and financial variables over the next few months or year (see, for example, Vissing-Jørgensen (2002), Greenwood and Shleifer (2014), and Cocco, Gomes, and Lopes (2019) for stocks, Piazzesi, Salomao, and Schneider (2015) for bonds, and Case and Shiller (2003), and Piazzesi and Schneider (2009) for real estate. In a similar vein, Andonov and Rauh (2020) study return expectations data for institutional investors from required financial reports). While the use of survey data is undoubtedly important, the noise inherent in this type of data underscores the need for alternative procedures to infer investors' beliefs.

We use a novel methodology that estimates investor beliefs from observed asset prices. Importantly, our method allows us to separately identify the beliefs and risk preferences of different types of investors. Most theories of financial markets assume that investors are homogeneous in terms of their risk preferences or their beliefs, or both. However, such homogeneity assumptions are difficult to reconcile with empirical evidence, such as the substantial variation in portfolio holdings across various groups of investors (see, e.g., Guiso, Halisassos, and Jappelli (2002), Calvet, Campbell, and Sodini (2007), Malmendier and Nagel (2011)), the huge volumes of trade in assets (see, e.g., Barberis and Thaler (2001)), and survey data on investors' beliefs (see, e.g., Greenwood and Shleifer (2014) and Giglio, Maggiori, Stroebel, and Utkus (2019)).

Our identification approach exploits the insight that different investor types choose different portfolios of securities. To separate preferences from beliefs, in our most basic specification we assume that the risk preferences of a given investor type can be summarized by a stochastic discount factor (SDF) that is an exponentially affine function of the return to that investor's wealth portfolio (this may be generalized to a set of factors that approximate the wealth portfolio return). The SDF coefficients measure the investors' risk appetites, and are allowed to vary across different investor types.

In our empirical implementation, we focus on two types of investors: those who allocate their wealth to large market capitalization stocks (hereafter referred to as Type I investors) and those who invest most of their wealth in small-growth firms, i.e. stocks with small market capitalization and low book-to-market equity ratios (hereafter referred to as Type II investors). Using these corner portfolios keeps the analysis tractable, and we show in robustness results that our results continue to hold with less extreme allocations, and including allocations to Treasury bills.

The beliefs of the Type I investors are crucial to understanding and predicting the aggregate equity market, the subject of long-standing and substantial research effort in financial economics. Large companies are the dominant component of the aggregate equity market portfolio, owing to the value-weighting scheme used for its construction. For example, as of May 2020 the largest 20% of publicly traded companies represented 83.3% of the total U.S. stock market capitalization. The Type II investor beliefs and expectations are also important. Despite the fact that the small-growth firms that they invest in only constitute a very small part of the market (In May 2020, the aggregate market value of the companies in the intersection of the smallest size and highest growth quintiles represented 0.2% of the U.S. stock market capitalization), the investment decisions of these firms are vitally important for economic growth and job creation (e.g., Decker, Haltiwanger, Jarmin, and Miranda (2014)). In fact, these stocks are often argued to be the public market counterpart of venture capital backed private start-up companies (see, e.g., Korteweg and Nagel (2016)), suggesting that the investors in these shares are likely to be quite different from large company stock investors. One possible reason is a preference for skewness, as the payoffs of small growth stocks are more lottery-like, similar to out-of-the money call options. Another possible contributing factor is their small size: The average market capitalization of the small-growth portfolio of U.S. stocks was $231 million as of May 2020, and none were larger than half a billion. As such, they do not attract much attention from large active asset managers, who tend to focus on larger stocks where they can invest a sizeable amount of money without adversely impacting prices or being subjected to disclosure rules (Koijen, Richmond, and Yogo (2020) show that, among institutional investors, active investors have the highest impact on asset prices). The conjecture that the investors in small-growth stocks are different from others is further strengthened by the observation that the very low historical average returns on these stocks have proven challenging for standard asset pricing models to explain (see, e.g., Fama and French (1993), Fama and French (2015)).

Our empirical results suggest strong evidence of heterogeneity in both beliefs and risk preferences across the two investor types. Specifically, the estimated market price of risk for the Type II investor is several fold smaller than that obtained for the Type I investor, suggesting that the former are substantially less risk averse than the latter. We also find significant heterogeneity in beliefs. The Type I investor believes that the expected stock market return is strongly countercyclical, steadily increasing through recessionary episodes to reach its peak around the end of the recession, before starting to decline as the economy enters the expansionary phase of the business cycle. Type II investors, on the other hand, believe that the expected market return is procyclical, i.e. high during expansions and declining through recessions. The correlation between the beliefs about the expected market return for the two types of investors is starkly negative at -39.4%. These findings are robust to the precise construction of the portfolios representing the large cap and small-growth market segments, as well as to the choice of sample period.

Our findings offer a potential resolution of a puzzling paradox in the existing literature. Standard rational expectations representative agent models (e.g., Campbell and Cochrane (1999) external habit formation model; Bansal and Yaron (2004) long run risks model; Barro (2006) and Wachter (2013) rare disasters models) imply countercyclical expected market returns. In contrast, investor surveys suggests that they believe that stock market returns are procyclical (Greenwood and Shleifer (2014)), a finding that has spurred interest in quantitative models of return extrapolation (e.g., Barberis, Greenwood, Jin, and Shleifer (2015)). Moreover, survey data across six different professional individual surveys are strongly positively correlated with each other as well as with observed behavior such as the number of IPOs. At the same time, the little evidence that is available on institutional investors' beliefs suggest that they may have mildly countercyclical beliefs. It may be conjectured that the individual and professional investors align with the investor types and their beliefs described above. Investors like those who primarily invest in large cap equity believe that expected market returns are countercyclical, consistent with the predictions of rational expectations representative agent models. Investors of the type that invest mostly in small-growth equity, on the other hand, believe that expected market returns are procyclical, consistent with the survey evidence.

Our econometric methodology is based on the non-parametric smoothed empirical likelihood (SEL) estimator, developed by Kitamura, Tripathi, and Ahn (2004). Given an SDF summarizing the risk preferences of a given investor type, the returns on a set of test assets (or portfolios) spanning the investors' opportunity set, and a specification of the conditioning set that these investors use to form their beliefs, our approach recovers the *joint conditional distribution* of the returns on the test assets and other variables entering the SDF, *as perceived by that investor type*, i.e. the investors' beliefs about the future. The approach also simultaneously delivers estimates of the SDF parameters that characterize the risk preferences of that investor type.

The SEL approach does not require any functional-form assumptions about the dynamics of beliefs or assumptions regarding investor rationality (or lack thereof). Instead, the method approximates the joint conditional distribution of the test asset returns and other variables entering the SDF with a multinomial distribution with support given by the available data sample. It then estimates the multinomial probabilities and the SDF parameters to maximize the likelihood of the data, enforcing the constraint that the test assets are perfectly priced, i.e. the conditional Euler equations are satisfied for these assets. Thus, the recovered beliefs and preferences are *price-consistent* – they are constructed such that each test asset at each point in time satisfies the conditional Euler equation restrictions for each investor type. Also, note that this approach does not impose apriori heterogeneity in either beliefs or preference parameters across investor types. Rather it offers a data-driven approach to identifying any potential heterogeneity.

The empirical implementation of the SEL method requires three inputs – the choice of the SDF's functional form, the set of test assets that the SDF is required to price, and the conditioning information set. We assume that the SDF of each investor type is exponentially affine in their total wealth portfolio. In other words, the SDF for the Type I investor is exponentially affine in a broad well-diversified portfolio of large cap stocks. For the Type II investor, on the other hand, the SDF is exponentially affine in a broad well-diversified portfolio of small-growth stocks. As for the choice of test assets, in order to avoid our results being contaminated by the idiosyncratic characteristics of individual stocks, we consider two test assets, a portfolio of large cap stocks and a portfolio of small-growth stocks. Finally, we assume that the investors' conditioning set consists of the lagged returns on the large cap and small-growth portfolios.

Finally, note that our methodology recovers the *entire conditional distribution* of the market return, as perceived by the two investor types. Therefore, we can ascertain all the dimensions along which there is heterogeneity in beliefs. Our results suggest that the heterogeneity arises primarily in the beliefs about the expected market return, but not in the volatility of the return which is perceived to be countercyclical by both groups of investors. This causes the perceived Sharpe ratio to be strongly countercyclical for the large cap investors, but strongly procyclical for the small-growth investors.

Our paper contributes to a growing literature that provides evidence, both theoretical and empirical, of investor heterogeneity and studies its implications for the macroeconomy and financial markets. Given the difficulty of disentangling beliefs from preferences, most of these studies only allow for heterogeneity along one dimension, i.e. in beliefs or preferences, but not both. For instance, Calvet, Campbell, Gomes, and Sodini (2019) allow for heterogeneity in preferences and estimate three preference parameters – the rate of time preference, the coefficient of relative risk aversion, and the elasticity of intertemporal substitution – for a cross-section of Swedish households. They assume that all households share the same beliefs, acknowledging that *"to the extent that any heterogeneity in beliefs exists, it will be attributed to heterogeneous preferences by our estimation procedure"*. A small literature aims to empirically recover heterogeneous beliefs: Meeuwis, Parker, Schoar, and Simester (2019) provide evidence of heterogeneity in beliefs by showing that households with different political party affiliation updated their portfolio holdings differentially after the surprise outcome of the 2016 US national election. Greenwood and Shleifer (2014) and Giglio, Maggiori, Stroebel, and Utkus (2019) show that survey data on investors' expectations also contain evidence of different types of investors with heterogeneous beliefs. In the theoretical models of Harrison and Kreps (1978), Scheinkman and Xiong (2003), Geanakoplos (2009), Atmaz and Basak (2018), investors are assumed to have heterogeneous beliefs while their preferences are assumed homogeneous. Moreover, these papers assume specific forms of beliefs heterogeneity that are made largely for tractability reasons rather than being optimally chosen in terms of empirical fit. To our knowledge, ours is the first attempt to directly estimate heterogeneous beliefs. Moreover, our methodology enables us to separately identify heterogeneous beliefs and risk preferences from observed asset prices. And, our approach does not require taking a stance on the origins or forms of beliefs heterogeneity or specific functional-form assumptions about the evolution of beliefs, or about investor rationality or lack thereof, lending robustness to potential misspecification.

The paper also contributes to a second strand of literature that uses an information-theoretic (or, relative-entropy minimizing) alternative to the standard generalized method of moments approach to address a variety of questions in economics and finance. Information-theoretic approaches were first introduced in financial economics by Stutzer (1995), Stutzer (1996) and Kitamura and Stutzer (1997) (see Kitamura (2006) for a survey of these methods). Subsequently, these approaches have been used to assess the empirical plausibility of the rare disasters hypothesis in explaining asset pricing puzzles (see, e.g., Julliard and Ghosh (2012)), construct diagnostics for asset pricing models (see, e.g., Almeida and Garcia (2012), Backus, Chernov, and Zin (2014)), construct bounds on the SDF and its components and recover the missing component from a candidate kernel (see, e.g., Almeida and Garcia (2016), Borovicka, Hansen, and Scheinkman (2016), Ghosh, Julliard, and Taylor (2016b), Sandulescu, Trojani, and Vedolin (2018)), and price broad cross sections of assets out of sample (see, e.g., Ghosh, Julliard, and Taylor (2016a)).

The paper closest to ours are Ghosh and Roussellet (2019) and Chen, Hansen, and Hansen (2020). The former also uses the SEL method to recover investors' beliefs from observed asset prices. Whereas Ghosh and Roussellet (2019) focus on recovering the beliefs of the representative investor in a complete markets setting, the present paper allows for heterogeneity in beliefs as well as in preferences across different investor types. Chen, Hansen, and Hansen (2020) present the theoretical underpinnings of the methododology used in Ghosh and Roussellet (2019). They show that information about investors' (potentially distorted) beliefs can be inferred from observed asset prices, given a candidate SDF and a statistical measure of divergence from a family that includes the empirical likelihood minimum distortion measure used in Ghosh and Roussellet (2019) and this paper.

The remainder of the paper is organized as follows. Section II presents our econometric methodology for the recovery of heterogeneous beliefs and preferences from observed asset prices. Section III describes the data and implementation choices. Section IV presents the estimated preferences and beliefs. Section V compares our recovered beliefs to survey expectations about the stock market. In Section VI, we compare the recovered beliefs with a proxy for the physical data generating process. Section VII discusses robustness, and Section VIII concludes with suggestions for future research.

II Methodology

In this section, we describe our framework and empirical methodology to recover the risk preferences and beliefs of different types of investors in financial markets. Throughout this section, uppercase letters denote random variables, while the corresponding lowercase letters represent particular realizations of these variables.

II.1 General Framework

The absence of arbitrage opportunities implies the existence of a strictly positive stochastic discount factor (SDF), denoted by M_{t+1}, such that the equilibrium returns $R^e_{i,t+1}$ of any traded asset i $(i = 1, 2, ..., I)$ in excess of the risk-free rate satisfy the conditional Euler equation:

$$\mathbb{E}^{\mathcal{P}} \left[M_{t+1} R^e_{i,t+1} | \underline{\mathcal{F}_t} \right] = 0, \tag{1}$$

where $\underline{\mathcal{F}_t} = \{\mathcal{F}_t, \mathcal{F}_{t-1}, \ldots\}$ denotes the investors' information set at time t, and $\mathcal{P}$ their beliefs about the future macroeconomic and financial outcomes conditional on $\underline{\mathcal{F}_t}$. In representative agent models based on rational expectations, for instance, M_{t+1} summarizes the risk preferences of the representative investor and $\mathcal{P}$ denotes her rational beliefs.

However, if financial markets are populated by heterogeneous investors, neither the risk preferences nor the beliefs may be unique across investor types. Specifically, there may exist different investor types l $(l = 1, 2, ..., L)$ that differ from one another, either in terms of their risk preferences, or in terms of their beliefs about the future, or both. If investor types differ in terms of their preferences, the SDF of the type–l investor can be expressed as $M^{(l)}$. If investor types differ in terms of their beliefs, the beliefs of the type-l investor can be denoted $\mathcal{P}^{(l)}$.

Thus, allowing for the possibility that different investor types have different risk preferences *and* beliefs, Equation (1) can be rewritten as:

$$\mathbb{E}^{\mathcal{P}^{(l)}}\left[M_{t+1}^{(l)}R_{i,t+1}^{e}|\underline{\mathcal{F}_t}\right]=0. \tag{2}$$

Our econometric procedure does not require us to take a stance on whether beliefs are rational. If the type-l investor is rational, then $\mathcal{P}^{(l)}$ in Equation (2) is the true (or, objective) probability measure that generates the observed data. If, on the other hand, the investor type has behavioral biases that make her beliefs deviate from rationality, then $\mathcal{P}^{(l)}$ denotes her subjective beliefs.

Finally, in order to disentangle beliefs from risk preferences, we need to make assumptions regarding the functional form of the SDF and the set of pricing factors, F,

$$M_{t+1}^{(l)}=M\left(F_{t+1}^{(l)};\gamma^{(l)}\right). \tag{3}$$

The parameters values $\gamma^{(l)}$, summarizing the risk preferences of investor type l, may vary by l and are to be estimated.

II.2 The Smoothed Empirical Likelihood Estimator

Our estimation uses the Smoothed Empirical Likelihood (SEL) approach developed by Kitamura, Tripathi, and Ahn (2004), which belongs to the family of non-parametric maximum likelihood estimators.

To provide some intuition behind the methodology, consider the type-l investor. Suppose that her SDF is fully known, such that the econometrician's problem reduces to only recovering her beliefs. In the absence of any parametric assumptions about beliefs, we would approximate beliefs with a multinomial distribution, with as many possible states as there are observation dates. Each state represents a particular joint realization of the pricing factors and the returns on the cross-section of assets in this investors' opportunity set. We require that the probability $p_t^{(l)}$ assigned to each state t is non-negative, and that the probabilities sum to one across all states, $t=1\ldots T$. In the absence of any further constraints, a standard maximum likelihood estimator would estimate each state to be equally likely to occur, i.e. $\widehat{\mathcal{P}}^{(l)}\equiv\widehat{p}_t^{(l)}=\frac{1}{T}$, for all t.

Now assume that we perform the same likelihood maximization, but enforcing the additional constraint that the unconditional Euler equation restrictions implied by Equation (2) are also satisfied:

$$\mathbb{E}^{\mathcal{P}^{(l)}}\left[M(F_t^{(l)};\gamma^{(l)})R_{i,t}^e\right] = 0 \Longleftrightarrow \sum_{t=1}^{T} p_t^{(l)} \cdot M(f_t^{(l)};\gamma^{(i)}) \cdot r_{i,t}^e = 0\,. \tag{4}$$

In order to satisfy the moment restrictions, the probabilities $\widehat{p}_t^{(l)}$ need to be adjusted relative to the $1/T$ benchmark. The empirical likelihood (EL) estimator of Owen (2001) does so by finding the probabilities that maximize the nonparametric log-likelihood of the observed data, subject to the constraint that the unconditional Euler equations are satisfied when evaluated under the estimated probability measure.

The SEL estimator relies on the same principle as the EL, but incorporates additional constraints through *conditional* Euler equation restrictions. To outline how this estimator works, assume that the investors' information set at time t can be summarized by a finite vector of random variables, $X_t \in \mathbb{R}^n$. States now represent the joint outcome of the pricing factors, the cross section of assets, and the conditioning information. Define $p_{j,k}$ as the conditional probability of state k being realized in the next period, given that the current state is j. Note that there are still as many states as there are observation periods.

The SEL estimator for the conditional probabilities, $\widehat{\mathcal{P}}^{(l)} \equiv \left\{p_{j,k}^{(l)}\right\}_{j,k=1}^{T}$, belongs to the simplex

$$\Delta := \cup_{j=1}^{T}\Delta_j = \cup_{j=1}^{T}\left\{(p_{j,1}^{(l)},...,p_{j,T}^{(l)}):\ \sum_{k=1}^{T} p_{j,k}^{(l)} = 1,\, p_{j,k}^{(l)} \geq 0\right\}.$$

For each state $j \in \{1,\dots,T\}$, given an admissible $\gamma^{(l)}$ parameter, the conditional probabilities are estimated as

$$\left\{\widehat{p_{j,\cdot}^{(l)}(\gamma^{(l)})}\right\} = \underset{(p_{j,\cdot}^{(l)})\in\Delta_j}{\arg\max} \sum_{k=1}^{T}\omega_{j,k}\log(p_{j,k}^{(l)}) \quad \text{s.t.} \quad \sum_{k=1}^{T} p_{j,k}^{(l)} \cdot M(f_k^{(l)};\gamma^{(l)}) \cdot r_{i,k}^e = 0, \tag{5}$$

where $p_{j,\cdot}^{(l)}$ denotes the $T-$dimensional vector of probabilities $(p_{j,1}^{(l)},...,p_{j,T}^{(l)})$, and $i = 1,2,...,I$ denotes the set of assets in the type-l investor's opportunity set. $\omega_{j,k}$ in Equation (5) are standard non-negative kernel weights used to smooth the objective function:

$$\omega_{j,k} = \frac{\mathcal{K}\left(\dfrac{x_j - x_k}{b_T}\right)}{\sum_{t=1}^{T}\mathcal{K}\left(\dfrac{x_j - x_t}{b_T}\right)}, \tag{6}$$

where $\mathcal{K}$ is a kernel function belonging to the class of second order product kernels, and the bandwidth b_T is a smoothing parameter.[1]

The intuition behind the estimator may be understood as follows. The objective function in Equation (5) is simply a 'smoothed' log-likelihood, with the constraints enforcing the conditional Euler equation restrictions in Equation (2). The conditioning information enters through the weights. Suppose that state j is realized at time t, i.e. $X_t = x_j$. The SEL estimator then focuses on a fixed neighbourhood around x_j, where the neighbourhood is defined in terms of the distance of other possible states from the current state, i.e. $|x_j - x_k|$, and not in terms of proximity in time. The estimator then assigns positive weights $\omega_{j,k}$ only to those states that lie within the fixed neighbourhood of the current state, with the exact values of the weights determined by the kernel function, the distance $|x_j - x_k|$, and the bandwidth parameter b_T (see Equation (6)). The states that lie outside the neighbourhood each receive a weight of zero and they are assigned a zero conditional probability. The conditional probability of each state with non-zero weight is then chosen so as to maximize the smoothed log-likelihood of the data subject to the constraint that the estimated conditional distribution satisfies the conditional Euler equations for the set of assets.

The solution to Equation (5) is analytical, given by:

$$\widehat{p_{j,k}^{(l)}(\gamma^{(l)})} = \frac{\omega_{j,k}}{1 + \widehat{\lambda_j(\gamma^{(l)})} \cdot M(f_k^{(l)}; \gamma^{(l)}) \cdot r_{i,k}^e}, \tag{7}$$

for all $j, k \in \{1, \ldots, T\}$. The $\widehat{\lambda_j(\gamma^{(l)})}$ are the Lagrange multipliers associated with the conditional Euler equation constraints, and solve the unconstrained problem

$$\widehat{\lambda_j(\gamma^{(l)})} = \arg\max_{\lambda_j \in \mathbb{R}^I} \sum_{k=1}^{T} \omega_{j,k} \log\left[1 + \lambda_j \cdot M(f_k^{(l)}; \gamma^{(l)}) \cdot r_{i,k}^e\right]. \tag{8}$$

Equations (7) and (8) show that the SEL procedure delivers a $(T \times T)$ matrix of probabilities $\left\{\widehat{p_{j,k}^{(l)}(\gamma^{(l)})}\right\}_{j,k=1}^{T}$ for a given value of the parameter $\gamma^{(l)}$. Each row $j : j = \{1, 2, ..., T\}$ contains the probabilities of moving to each of the T possible states $k : \{k = 1, 2, ..., T\}$ in the next period, conditional on state j having been realized in the current period. Therefore, the SEL approach recovers the *entire conditional distribution* of the data. This $(T \times T)$

[1] $\mathcal{K}$ should satisfy Assumption 3.3 in Kitamura, Tripathi, and Ahn (2004), that is restated here for convenience. For $X = (X^{(1)}, X^{(2)}, ..., X^{(n)})$, let $\mathcal{K} = \prod_{s=1}^{n} k(X^{(s)})$. Here $k : \mathbb{R} \to \mathbb{R}_+$ is a continuously differentiable p.d.f. with support $[-1, 1]$. k is symmetric about the origin, and for some $\alpha \in (0, 1)$ is bounded away from zero on $[-a, a]$. In theory, b_T is a null sequence of positive numbers such that $Tb_T \to \infty$. See Assumption 3.7 in Kitamura, Tripathi, and Ahn (2004) for additional restrictions on the choice of b_T.

probability matrix represents the *beliefs* of the type-l investor: $\widehat{\mathcal{P}^{(l)}} = \left\{ \widehat{p_{j,k}^{(l)}(\gamma^{(l)})} \right\}_{j,k=1}^{T}$. Note that these recovered beliefs are consistent with observed asset prices, i.e. they satisfy the conditional Euler equations for the chosen cross section of assets.

Note that the SEL approach recovers the type-l investor's beliefs, i.e. the conditional distribution of the data as perceived by her, without the need for any parametric assumptions about the shape of the distribution. It does so by approximating the conditional distribution, for each possible value of the state, as a multinomial on the observed data sample. It may seem that this requires the estimation of $T \times T$ conditional probabilities, given a sample size of only T. However, the number of parameters that the approach needs to estimate is only $T \times I$, where I denotes the number of test assets. Specifically, for each date (or, state), the SEL procedure only requires the estimation of the vector of Lagrange multipliers associated with the conditional Euler equation restrictions. Therefore, for each date, the number of parameters to be estimated is the same as the number of test assets that the SDF is asked to price (see Equations (7) and (8)).[2]

A useful point of comparison is that without the pricing restrictions, Equation (7) simplifies to $\widehat{p_{j,k}^{(l)}(\gamma^{(l)})} = \omega_{j,k}$. This is simply a standard kernel density estimate of the conditional distribution. Imposition of the pricing restrictions may thus be viewed as distortions in beliefs (relative to the physicial probabilities) necessary to satisfy the pricing restrictions. In Section VI we show that the SEL procedure produces the probability measure that satisfies the Euler restrictions while deviating as little as possible from the physical measure.

We next turn to the recovery of the risk preferences of investor type l. Note that the notation $\left\{ \widehat{p_{j,k}^{(l)}(\gamma^{(l)})} \right\}$ emphasizes that the estimated beliefs are a function of the SDF parameter $\gamma^{(l)}$. The SEL method also allows to estimate it. Specifically, the SEL estimator of $\gamma^{(l)}$ is defined as:

$$\widehat{\gamma^{(l)}}^{SEL} = \underset{\gamma^{(l)} \in \Theta}{\operatorname{argmax}} \quad \underbrace{\sum_{j=1}^{T} \sum_{k=1}^{T} \omega_{j,k} \log(\widehat{p_{j,k}^{(l)}(\gamma^{(l)})})}_{SEL(\gamma^{(l)})} . \tag{9}$$

In other words, the approach searches for the value of $\gamma^{(l)}$ that maximizes the profile log-likelihood over the admissible parameter set, a procedure reminiscent of parametric maximum likelihood estimation of a finite-dimensional vector of parameters.

Appendix A describes the asymptotic and finite-sample properties of the SEL estimator, both in the case when the conditional Eurler equation (2) is correctly specified and when it

[2]This dramatic reduction in the dimensionality of the optimization problem is achieved because the SEL estimator is the solution to a convex optimization problem, and, therefore, the Fenchel duality applies (see, e.g., Borwein and Lewis (1991)).

is misspecified.

II.3 Simulation Evidence

Ghosh and Roussellet (2019) present extensive simulation evidence on the performance of the SEL estimator of the SDF parameters and beliefs in representative agent economies. In the interests of brevity, we briefly summarize their main findings here and refer the reader to Ghosh and Roussellet (2019) for a more detailed exposition.

The authors first consider the case of correctly specified models, i.e. scenarios where the SDF summarizing the investors' risk preferences is correctly specified and investors have rational beliefs. Through three simulated economics – that include the standard Consumption-CAPM (CCAPM) with power utility and i.i.d lognormal aggregate consumption growth rate, the Campbell and Cochrane (1999) external habit formation model, and the Bansal and Yaron (2004) long run risks model – they show that the SEL method accurately recovers the utility curvature parameter as well as the physical distribution of the data in samples of the same length as the historical post war period.

They also present results in the presence of SDF misspecification, i.e. when the true model that generates the data corresponds to the external habit or long run risks model but the econometrician erroneously applies the SEL approach to a power utility SDF. They show that, even in these scenarios, the SEL recovers reasonable accurately the true physical distribution of the data, albeit at substantially larger values of the utility curvature parameter relative to its true values.

Finally, they present results when the SDF is correctly specified but investors' beliefs are distorted relative to the physical data generating process (DGP). Specifically, they consider the standard CCAPM economy described above, but assume that investors are pessimistic and act as if consumption growth had a lower mean and/or higher volatility relative to the physical DGP. They show that, in such a scenario, the SEL accurately recovers the distorted beliefs of investors. The rational for this finding is that the Euler equation constraints, that are required to be satisfied by the SEL estimator, are satisfied under the distorted beliefs but not under the true physical DGP.

Overall, the results suggest that the SEL approach is successful at recovering investors' beliefs with empirically realistic sample sizes. This lends support to extend the use of this methodology to recover heterogeneous beliefs.

III Implementation

To implement the SEL estimator we need to make three choices: i) the functional forms of investors' SDFs that summarize their risk preferences; ii) portfolios to represent the returns in each market segment that is to be priced (the "test assets"), and; iii) the conditioning information set.

In standard asset pricing models, an investor's SDF is a function of her consumption or wealth. In keeping with this tradition, we assume that the SDF of a representative type-l investor is exponentially affine in the return on her total wealth portfolio:

$$M\left(F_{t+1}^{(l)};\gamma^{(l)}\right) = e^{-\gamma^{(l)}\log\left(R_{W,t+1}^{(l)}\right)}, \tag{10}$$

where $R_{W,t+1}^{(l)}$ is the investor's gross return on total wealth. The above SDF can be micro-founded by, for instance, investors having exponential utility, with the coefficient of absolute risk aversion (CARA) given by $\gamma^{(l)}$.

If the portfolio holdings of different investor types were known, then these could be used for R_W in Equation (10) to recover their risk preferences and beliefs. Lacking such information, we have to make assumptions about investors' portfolio holdings. To strike a compromise between econometric tractability and realistic portfolio choices, we assume that each investor type optimally allocates her wealth between three market segments: (a) a portfolio of large market capitalization stocks (akin to the S&P 500), (b) a portfolio of small market capitalization and growth stocks, and (c) short term Treasury bonds. Thus, the return on the type-l investor's wealth portfolio is $R_{W,t+1}^{(l)} = x_{B,t}^{(l)}R_{B,t+1} + x_{SG,t}^{(l)}R_{SG,t+1} + (1 - x_{B,t}^{(l)} - x_{SG,t}^{(l)})R_{F,t}$, where $x_{B,t}^{(l)}$ and $x_{SG,t}^{(l)}$ are the proportions allocated to large cap equity and small-growth equity, respectively, at time t.[3] Each investor type may choose a different $\left(x_{B,t}^{(l)},\ x_{SG,t}^{(l)}\right)$ because their beliefs and/or preferences may differ from those of other investor types. A type-l investor's beliefs and preferences, thus, need to satisfy the Euler equations:

$$\mathbb{E}^{\mathcal{P}^{(l)}}\left[\left(e^{-\gamma^{(l)}\log\left(x_{B,t}^{(l)}R_{B,t+1}+x_{SG,t}^{(l)}R_{SG,t+1}+(1-x_{B,t}^{(l)}-x_{SG,t}^{(l)})R_{F,t}\right)}\right)R_{i,t+1}^{e}|\underline{\mathcal{F}_t}\right] = 0, \tag{11}$$

for all test assets $i = 1 \ldots I$. We use the SEL approach to recover the investor's risk aversion parameter, $\gamma^{(l)}$, and beliefs, $\mathcal{P}^{(l)}$.

To help with the interpretation of the findings, we mainly focus our attention on two special investor types. First, a Type I investor for whom $\left(x_{B,t}^{(l)},\ x_{SG,t}^{(l)}\right) = (1,\ 0)$. Such an investor allocates all of her wealth to large market cap equity. Second, at the other end

[3]While we focus on the choice between two risky equity portfolios and one risk free asset, the methodology is quite general and can be extended to include additional risky assets. We leave this for future research.

of the spectrum, a Type II investor with $\left(x_{B,t}^{(l)},\, x_{SG,t}^{(l)}\right) = (0,\, 1)$, who allocates all of her wealth to small-growth stocks. We show in robustness tests that relaxing these extreme weights, including allowing for an allocation to the risk-free asset does not change the results qualitatively.

Note that although different investor types with different beliefs and/or risk preferences optimally choose different asset mixes from one another, each investor's beliefs and preferences satisfy the Euler equations for *all* the assets in the mix, not just the assets that she optimally chooses to invest most (or even all) of her wealth in. This ensures that, given an investor's choice of optimal portfolio, her recovered beliefs and preferences satisfy the Euler equations for all assets in her opportunity set, i.e. her first-order conditions.

For R_B we use the excess return on a portfolio of large market capitalization U.S. stocks and for R_{SG} we use the excess return on U.S. small-growth stocks, characterized by small market capitalizations and low values of the book-to-market-equity ratio. Specifically, for segment B, we use the average return of the five highest market capitalization portfolios of the Fama-French 25 portfolios, formed from the intersection of five size-sorted and five book-to-market-equity (BE/ME) sorted portfolios (Fama and French (1993)). We refer to this portfolio as B^{FF25}. For robustness, we also present results for the average return on the three big-size portfolios of the Fama-French 6 portfolios, formed from the intersection of two size-sorted and three BE/ME-sorted portfolios (B^{FF6}). Similarly, we consider two alternative portfolios summarizing segment SG: the portfolio of the smallest size, highest growth stocks from the Fama-French 25 and Fama-French 6 portfolios (SG^{FF25} and SG^{FF6}), respectively. The proxy for the risk free rate is the one-month Treasury bill rate. Monthly returns on all these portfolios are obtained from Kenneth French's data library, and compounded within each quarter to obtain quarterly returns.

Our baseline sample covers the period 1972:Q1 – 2018:Q4. The start date of 1972:Q1 is chosen to coincide with the introduction of the NASDAQ exchange, where the majority of the small cap and growth stocks trade. For robustness, we also present results with monthly data over this period as well as with quarterly data over the longer post war period 1947:Q1 – 2018:Q4.

For the set of test assets, R_i^e, we use the same R_B and R_{SG} portfolio returns. Both Type I and Type II investors are thus expected to price the returns on the B and SG portfolios, and the Euler equation (11) holds for both test assets and for both investor types.

For the conditioning information set, which captures the variation in the investment opportunity set over time, we use lagged returns on the large market cap and small-growth portfolios.

Finally, to compute the kernel density weights in Equation (6), we use the Epanechnikov kernel with bandwidth b_T equal to three times the standard deviations of the conditioning variables.[4]

IV Empirical Results: Preferences and Beliefs

In this section, we present the estimated risk aversion coefficients (the SDF parameter $\gamma^{(l)}$) and the beliefs of the Type I and Type II investors, described in Section III. Recall that the Type-I investor allocates to large market capitalization equity, whereas the Type-II investor primarily invests in small-growth equity. Note that, by construction, the recovered preferences and beliefs of each investor type are consistent with the observed prices of both the large cap and small-growth portfolios. Specifically, the preferences and beliefs of each investor type produce a time series of conditional pricing errors for the above two portfolios that identically equals zero.[5]

Section IV.1 presents the estimates of the risk aversion parameter for the two investor types, and Section IV.2 presents their recovered beliefs.

IV.1 Recovered Risk Preferences

The SDF parameter $\gamma^{(l)}$ captures the price of (or, the compensation for bearing) risk, required by the type-l investor, $l = \{I, II\}$. More specifically, as described in Section III, our chosen functional-form of the SDF can be motivated by investors having exponential utility over their returns on total wealth, with $\gamma^{(l)}$ denoting the coefficient of absolute risk aversion of the type-l investor.

Table 1 presents the estimates of $\gamma^{(l)}$ for the two investor types, for our baseline sample period $1972Q1$–$2018Q4$. Consider first Rows 1–2 that present the results for the Type I investor. In Row 1, the large cap and small-growth test portfolios are constructed from the 25 FF portfolios, whereas in Row 2 they are obtained from the 6 FF portfolios. In Column 2, the conditioning set consists of the one-quarter lagged returns on the large cap and small growth portfolios. In Column 3, on the other hand, the conditioning set consists

[4] The results are robust to alternative choices of the kernel function, and to a smoothing parameter equal to four standard deviations of the conditioning variables. These results are omitted for brevity and are available from the authors upon request.

[5] The conditional pricing error of an asset or portfolio i in period t, implied by the recovered preferences and beliefs of investor type-l is defined as:

$$\widehat{\mathbb{E}^{\mathcal{P}^{(l)}}}\left[\left\{e^{-\gamma^{(l)}\log(R^{(l)}_{W,t+1})}\right\}R_{i,t+1}|X_t\right] = \sum_{k=1}^{T}\widehat{p}^{(l)}_{t,k}\cdot\left\{e^{-\widehat{\gamma^{(l)}}\log(r^{(l)}_{W,k})}\right\}r_{i,k}\,. \tag{12}$$

of the averages of the returns on the large cap and small growth portfolios over the last four quarters.

The $\gamma^{(I)}$ estimate is 1.8 and 2.0, respectively, in Rows 1 and 2, using the one-quarter lagged returns on the test assets as the conditioning set (Column 2). Using four lags of the test assets' returns in the conditioning set (Column 3) give very similar estimates of 2.1 and 2.4, respectively. All of these parameters are highly statistically significant.

Table 1 – Estimates of Preference Parameter, $1972Q1 - 2018Q4$

	$\gamma^{(i)}$		LR Test ($H_0 : \gamma^{(i)} = \gamma^{II}$)	
	COND. SET 1	COND. SET 2	COND. SET 1	COND. SET 2
TYPE-I: B^{FF25}, SG^{FF25}	1.8 (0.89)	2.1 (0.84)	3.59 [.058]	4.62 [.032]
TYPE-I: B^{FF6}, SG^{FF6}	2.0 (0.89)	2.4 (0.92)	2.55 [.110]	3.26 [.071]
TYPE-II: B^{FF25}, SG^{FF25}	0.2 (0.46)	0.3 (0.48)	-	-
TYPE-II: B^{FF6}, SG^{FF6}	0.7 (0.56)	0.8 (0.57)	-	-

The table reports the point estimates of the SDF parameter $\gamma^{(l)}$, along with the asymptotic standard errors in parentheses below, for the Type-I investor (Rows 1-2) and Type-II investor (Rows 3-4). Type I investors allocate all of their wealth to large market capitaliztion stocks. Type II investors invest all their wealth in small-growth stocks. The estimates are obtained using the SEL method, with the SDF exponentially affine in the large cap portfolio return in Rows 1 and 2 and in the small-growth portfolio return in Rows 3 and 4. The conditioning set consists of the last quarter's large cap and small-growth returns (Cond. Set 1; Column 2) and the averages of the last four quarters' large cap and small-growth returns (Cond. Set 2; Column 3). Columns 4 and 5 report likelihood ratio tests, with the associated p-values in square brackets below, of the null hypothesis that $\gamma^{(l)}$ is equal across the two investor types. The test statistic has a $\chi^2_{(1)}$-distribution under the null. The sample is quarterly, covering the period $1972Q1 - 2018Q4$.

Consider next Rows 3–4 that present the results for the Type II investor. The test assets in Rows 3 and 4 are constructed from the 25 FF and 6 FF portfolios, respectively, i.e. they are identical to those in Rows 1 and 2, respectively. Row 3, Column 2 shows that, using the one-quarter lagged test asset returns as conditioning information, the $\gamma^{(l)}$ estimate for the Type-II investor is 0.2 – nine times smaller than the 1.8 estimate obtained for the Type-I investor in Row 1, Column 2. Changing the conditioning information to the average test asset returns over the most recent four quarters yields a $\gamma^{(II)}$ of 0.3, once again an order of magnitude smaller than that obtained for the Type-I investor. To see whether the results for the Type-II investors are driven by the micro cap growth stocks, Row 4 presents the gamma estimates when the two test portfolios are constructed from the 6 FF portfolios. The estimate of $\gamma^{(l)}$ is 0.7 for the Type-II investors, less than half the value of 1.8 for the Type-I investors, when the one-quarter lagged test asset returns are included in the conditioning information. The results for the alternative information set are nearly identical.

Columns 4-5 present likelihood ratio (LR) tests of the null hypothesis that $\gamma^{(l)}$ is equal across the two investor types. Specifically, Row 1, Column 4 presents the test statistic for the test $H_0 : \gamma^{(I)} = \gamma^{(II)}$, against the alternative $H_1 : \gamma^{(I)} \neq \gamma^{(II)}$. In three out of the four scenarios – the coarser and finer portfolio sorting to obtain the large cap and the small growth portfolios, and the two different choices of the conditioning set – the null hypothesis is rejected at the 10% level of significance. In the single scenario where the LR test fails to reject the null, the test statistic has a pvalue of 11.0%, only slightly bigger than the 10% threshold.

Overall, our results suggest that the Type-II investors require a smaller price of risk compared to the Type-I investors. This is indicative of the former group being less risk averse compared to the latter. And the difference in risk appetites between the two groups of investors is statistically and economically significant.

IV.2 Recovered Beliefs

In addition to the risk preferences, the SEL approach also delivers estimates of the beliefs of each investor type l, denoted by $\widehat{\mathcal{P}^{(l)}} = \left\{\widehat{p_{j,k}^{(l)}}\right\}_{j,k=1,\dots,T}$. Recall that, in this context, beliefs refer to the joint distribution of the returns on the large cap portfolio (akin to aggregate equity market indices such as the S&P 500) and the small-growth portfolio. Armed with these beliefs, it is easy to compute the conditional moments of returns, as perceived by the type-l investor, for different values of the conditioning state. For instance, the expected market return at time $t+1$, conditional on the information available at time t, is given by:

$$\widehat{\mathbb{E}^{\mathcal{P}^{(l)}}}\left[R_{m,t+1}|X_t\right] = \sum_{k=1}^{T} \widehat{p_{t,k}^{(l)}} \cdot r_{m,k}\,. \tag{13}$$

Other moments, such as the conditional volatility and skewness of the market return and its Sharpe ratio, can be similarly computed. Since, as we will see below, the estimated beliefs differ across the two investor types, i.e. $\widehat{\mathcal{P}^{(I)}} \neq \widehat{\mathcal{P}^{(II)}}$, they also perceive the conditional moments of the market return to be different.

We first present the recovered beliefs about the aggregate equity market return for the two investor types. Figure 1 presents the time series of the conditional mean (Panel A), volatility (Panel B), Sharpe ratio (Panel C), and skewness (Panel D) of the market return. The solid black (red-dashed) line in each panel presents the moments for the Type I (Type II) investor.

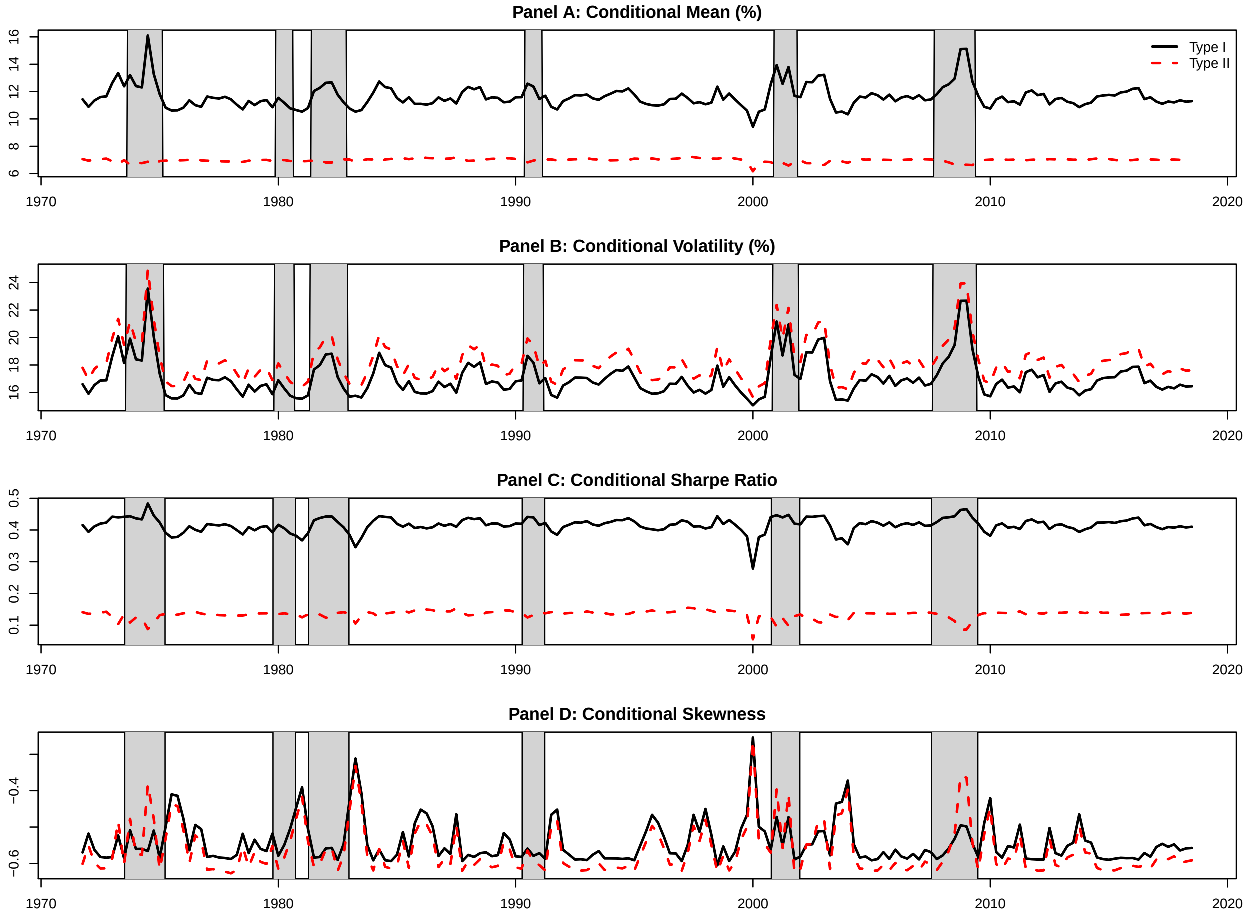
Panel A shows that the expected market return is strongly time-varying for the Type I investor – it ranges from 9.4% to 16.1% over the sample period. The expected market return is also strongly countercyclical, rising sharply during recessionary episodes. The correlation between the expected market return and a recession dummy is 51.6%. The beliefs of the Type II investor appear flat in Panel A because both the level and the variability of the expected market returns are a lot higher for the Type I investor compared to the Type II investor. Figure 3, that presents the beliefs of the Type II investor alone, shows that the expected market returns differ starkly across the two investor types, along at least two important dimensions. First, the average level of the expected market return is much smaller for the Type II investor compared to the Type I investor (7.0% versus 11.6%). This is, perhaps, not surprising given that the risk aversion parameter of the former is estimated to be substantially smaller than that of the latter (0.3 versus 2.1). However, this is not the only dimension of the beliefs heterogeneity across the two investor types. The beliefs about the expected market return, formed by the Type II investor, is strongly procyclical, with a correlation of −44.8% with the recession dummy. Note that this is the opposite of the beliefs recovered for the Type I investor.

Panel B shows that the perceived conditional volatility of the market return is very similar across the two investor types. The volatility is strongly countercyclical, rising sharply during NBER-designated recession periods. Specifically, for the Type I investor, the volatility ranges from 15.1% to 23.6%, with a mean of 17.0% and a correlation of 54.7% with the recession dummy. For the Type II investor, the volatility ranges from 15.7% to 25.0%, with a mean of 18.2% and a correlation of 53.6% with the recession dummy.

Panel C shows that the perceived Sharpe ratios are very different between the two investor types. The first difference lies in the level of the sharpe ratio – the Type I investor's beliefs imply an average Sharpe ratio of 0.42 while the beliefs of the Type II investor imply a much smaller average Sharpe ratio of only 0.13. The second difference lies in the perceived dynamics of the Sharpe ratio. The more pronounced countercyclical variation in the conditional mean relative to the volatility for the Type I investor leads to a countercyclical Sharpe ratio – its correlation with the recession dummy is 37.9%. For the Type II investors, on the other hand, the pro-cyclical expected market return, coupled with the countercyclical volatility of the market return, leads the perceived Sharpe ratio to be strongly pro-cyclical with a correlation of −44.5% with the recession dummy.

Finally, Panel D shows that the tail risk of the market, as captured by its conditional skewness, is perceived to be similar by the two investor types. Specifically, for the Type I investor, the skewness ranges from −0.61 to −0.25, with a mean of −0.54 and a small correlation of −4.7% with recessions. For the Type II investor, the skewness ranges from

Figure 1 – Beliefs About Market Return



Notes: The figure plots the time series of the conditional mean (Panel A), volatility (Panel B), Sharpe ratio (Panel C), and skewness (Panel D) of the market return, as perceived by the Type I investor (solid black like) and Type II investor (red-dashed line). Types are as described in Table 1. Shaded areas denote NBER designated recession periods. The conditional moments are obtained using the estimated SEL distributions. The pricing kernel is exponentially affine in the large cap portfolio return for the Type I investor and in the small-growth portfolio return for the Type II investor. The test assets consist of the excess returns on the large cap and small growth portfolios. The conditioning set consists of the averages of the last four quarter's large cap and small-growth portfolio returns. The sample is quarterly covering the period 1972:Q1-2018:Q4.

-0.63 to -0.26, with a mean of -0.57 and a small correlation of 16.3% (albeit opposite in sign compared to that obtained for the Type I investor) with the recession dummy.

In Figure 1, the large cap and small-growth portfolios were constructed from the 25 FF portfolios, and the averages of the past four quarter's returns on these two portfolios were used as conditioning variables. We also show that the recovered beliefs about the stock market for the two investor types are very similar when the large cap and small growth portfolios are constructed from the coarser 6 FF portfolios, pointing to the robustness of the methodology.[6]

Note that the recovered patterns of time-variation in the conditional moments of the market return for the Type I investors are in line with the implications of rational expectations based representative agent models. The prominent paradigms in the literature (see, e.g., Campbell and Cochrane (1999) external habit formation model, Bansal and Yaron (2004) long run risks model, Barro (2006) and Wachter (2013) rare disasters model) all imply countercyclical expected returns and Sharpe ratio for the market. A potential concern in interpreting our results is the sensitivity of the recovered beliefs to the specification of the SDF that summarizes investors' risk preferences. Recall that we have assumed that the SDF for each investor type is exponentially affine in their return on wealth. However, a number of alternative specifications could have been assumed. In this context, Ghosh and Roussellet (2019) show that very similar beliefs about the aggregate stock market are recovered for a variety of different specifications of the SDF that include all the prominent paradigms mentioned above. Therefore, to summarize, our results suggest that, for investors who allocate all or most of their wealth to broad diversified portfolios dominated by large market capitalization stocks, then regardless of the exact specification of these investors' risk preferences as summarized by an SDF, the recovered beliefs imply strong countercyclical variation in the expected market returns and its Sharpe ratio.

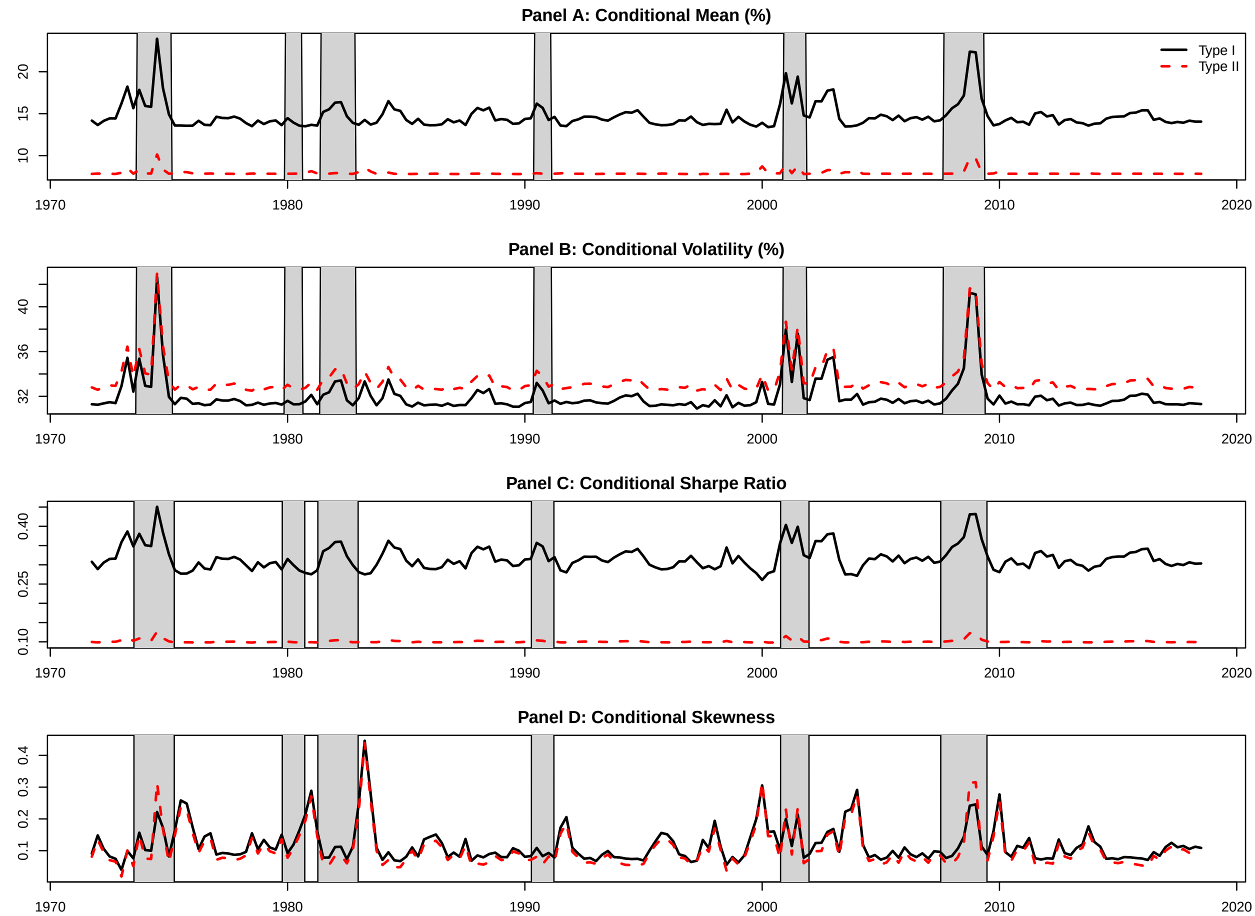
We argue in Section V that the beliefs recovered for the Type II investor, although starkly different from those of the Type I investor, are in line with the reported beliefs in surveys of professional individual investors (see, e.g., Greenwood and Shleifer (2014)).

So far, we focused on the beliefs about the aggregate equity market return formed by different investors types. We now turn to a comparison of the beliefs about small-growth stocks formed by the two types of investor. These beliefs are presented in Figure 2. Panels *A*-*D* plot the beliefs of the Type I (solid black line) and Type II (red-dashed line) investors, about the mean, volatility, Sharpe ratio, and skewness, respectively, of the small-growth portfolio.

Panel *A* reveals that the main difference in the beliefs about the expected returns on the small-growth portfolio across the two investor types lies in their relative levels. Specifically, for the Type I investor, the expected return ranges from 13.4% to 23.9%, with a mean of 14.7%. In contrast, for the Type II investor, the expected returns are roughly half of those formed by the Type I investor, varying from 7.7% to 10.1%, with a mean of 7.9%. However, for both investor types, the expected returns are countercyclical (albeit more strongly so for the Type I investor) – the correlations between the expected market returns and the recession dummy are 54.1% and 37.3%, respectively, for the Type I and Type II investors.

[6]These results are omitted for brevity and are available from the authors upon request.

Figure 2 – Beliefs About Small-Growth Portfolio Return

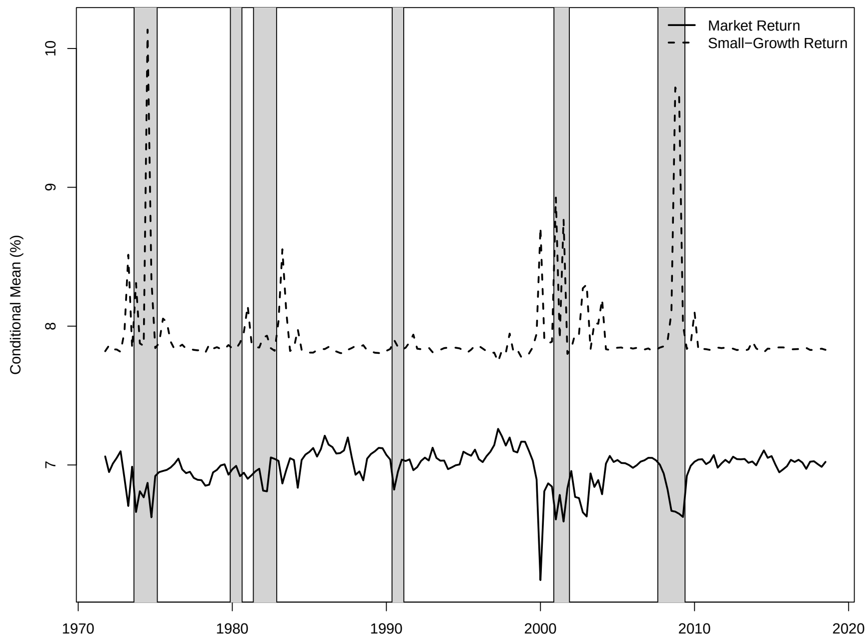


Notes: The figure plots the time series of the conditional mean (Panel A), volatility (Panel B), Sharpe ratio (Panel C), and skewness (Panel D) of the small-growth portfolio return, as perceived by the Type I investor (solid black like) and Type II investor (red-dashed line). Types are as described in Table 1. Shaded areas denote NBER designated recession periods. The conditional moments are obtained using the estimated SEL distributions. The pricing kernel is exponentially affine in the large cap portfolio return for the Type I investor and in the small-growth portfolio return for the Type II investor. The test assets consist of the excess returns on the large cap and small growth portfolios. The conditioning set consists of the averages of the last four quarter's large cap and small-growth portfolio returns. The sample is quarterly covering the period 1972:Q1-2018:Q4.

Panel *B* shows that the two investor types have similar beliefs about the volatility of returns on the small-growth portfolio. Specifically, both investor types perceive the small-growth sector to be substantially more volatile than the overall market, and the volatility to be strongly countercyclical.

The countercyclical mean and volatility, with more pronounced time-variation in the mean relative to the volatility, causes the Sharpe ratio of the small-growth portfolio return

Figure 3 – Beliefs of Type II Investors



Notes: The figure plots the beliefs of Type II investors, who invest their wealth in small-growth stocks, about the expected market return (solid line) and expected return on small-growth stocks (dashed line). Shaded areas denote NBER designated recession periods. The conditional moments are obtained using the estimated SEL distributions. The pricing kernel for Type II investors is exponentially affine in the small-growth portfolio return. The test assets consist of the excess returns on the large cap and small growth portfolios. The conditioning set consists of the averages of the last four quarter's large cap and small-growth portfolio returns. The sample is quarterly covering the period 1972:Q1-2018:Q4.

to be strongly countercyclical for both investor types – the correlations between the Sharpe ratio and the recession dummy are 53.1% and 51.9%, respectively, for the Type I and Type II investors. The greater risk tolerance of the Type II investor results in them requiring a much smaller Sharpe ratio than the Type I investor – the average Sharpe ratio for the two investor types are 0.32 and 0.10, respectively.

Finally, Panel *D* shows that the two groups of investors have similar beliefs about the skewness of returns on the small-growth portfolio. In particular, both investors perceive the skewness to be close to zero on average, and become positive during recessionary episodes.

To summarize, the results suggest two major dimensions of beliefs heterogeneity between the two investor types. First, the Type I investor perceives the expected aggregate equity market returns and its Sharpe ratio to be countercyclical whereas the Type II investor perceives them to be procyclical. Second, the former investors form much higher estimates of the expected returns and Sharpe ratios on both the aggregate market as well as the small-growth segment compared to the latter investors. The estimates of the conditional volatility and skewness, on the other hand, are very similar across the two investor types for both the market and small-growth portfolios.

Overall, the results of this section suggest that the two investors types differ substantially from each other, both in terms of their risk appetites as well as their beliefs about the future. The next two sections characterize the dimensions of this heterogeneity in further detail.

V Comparison with Survey Data

In Section IV, we recovered the beliefs about the aggregate stock market, of two types of investors. Our identification strategy relied on the insight that investors with different beliefs and/or risk preferences optimally choose different portfolio holdings. While this offers an elegant framework to recover beliefs, it is admittedly reliant on assumptions about the functional form of the SDF summarizing investors' risk preferences. This raises the potential concern as to whether what we are recovering are indeed the beliefs of investors or rather do they represent omitted risk factors in the SDF.

Fortunately, there is also more direct data available on investors' beliefs. Specifically, several surveys have been conducted in the US, dating back to as early as 1987, that ask institutional and professional individual investors about their beliefs about the stock market. In this section, we present evidence that the beliefs that we recover from asset prices are strongly correlated with survey data on investors' expectations about the stock market. More importantly, recall that our findings suggest substantial heterogeneity in beliefs across different cohorts of investors. Specifically, we find that investors who allocate most of their wealth in large market cap equity have countercyclical beliefs about the expected market return. To the contrary, investors who mostly allocate to small growth stocks believe that the expected market return is procyclical. In this section, we show that the above type of beliefs heterogeneity is also consistent with survey data on investors' beliefs.

We first consider surveys of individual investors. Greenwood and Shleifer (2014) study six such surveys: (i) the Gallup survey, conducted monthly between 1996-2012 with occasional gaps, (ii) the Graham-Harvey (GH) survey of Chief Financial Officers of major US corporations, conducted quarterly since October 2000, (iii) the American Association (AA) of Individual Investors Investor Sentiment Survey, administered weekly since 1987, (iv) Robert Shiller's survey of wealthy individual investors (Shiller Ind), available sporadically between 1999 and July 2001 and monthly thereafter, (v) the Investors' Intelligence (II) newsletter expectations, available since 1963, and (vi) the Michigan survey, available over the period 2000-2005, that asked respondents about their beliefs about the stock market.

The authors' show that investors' beliefs about the stock market are positively correlated across these surveys – the pairwise correlations between the surveys (i)-(v) above vary from 0.39 to 0.77 and are all statistically significant. The Michigan survey has a statistically significant positive correlation with only two of the other five surveys, but note that this survey is only available for a very short time period 2000-2005. Greenwood and Shleifer (2014) also show that the reported beliefs in these surveys are strongly correlated with mutual fund flows and the number of IPOs. They use these findings to argue that the survey data contain valuable information about investors' expectations and should not be discarded as noisy and useless.

More importantly, Greenwood and Shleifer (2014) point out that the above surveys suggest that investors' beliefs about the aggregate stock market are strongly pro cyclical – their expectations about the stock market are high during expansions and low during the recessionary phase of the business cycle. Indeed, the correlation with a NBER recession dummy variable is -51.6% for the Gallup survey, -36.8% for the GH survey, and -36.3% for the AA survey. The Shiller Ind survey is the only individual investor survey that has a correlation close to zero (1.4%) with the recession dummy.[7] As an illustration, Figure 4 plots the time series of investors' expectations from the Gallup survey (solid black line), the GH survey (red-dashed line), the AA survey (blue-dotted line) and the Shiller Ind survey (green dashed-dotted line). The procyclical nature of the expectations is evident from the first three surveys.

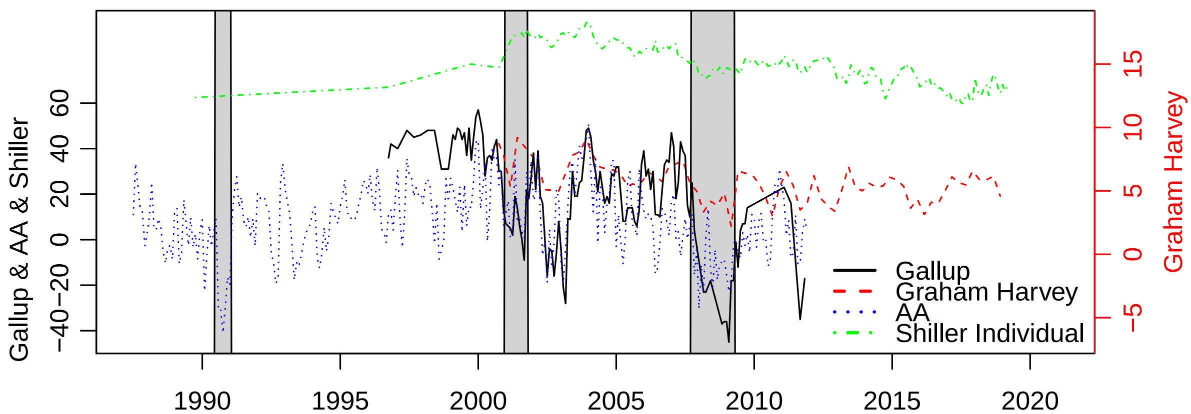
Note that rational expectations representative agent models (see, e.g., Campbell and Cochrane (1999) external habit formation model, Bansal and Yaron (2004) long run risks model, Barro (2006) and Wachter (2013) rare disasters model) all imply countercyclical expected market returns, whereby the risk averse representative investor requires a higher compensation for bearing market risk during bad times. Greenwood and Shleifer (2014), therefore, argue that the evidence on investors' expectations in survey data is inconsistent with rational expectations representative agent models.

However, a comparison of the survey expectations about the stock market with our recovered beliefs immediately suggests that the professional individual investors covered in these surveys are akin to the Type II investor, i.e. investors that allocate heavily to small-growth type of assets. Recall that our recovered beliefs about the stock market for the Type II investor suggest pro cyclical expected market returns, consistent with the reported beliefs in the above individual investor surveys.

We next turn to surveys that cover institutional (rather than individual) investors. Specifically, we consider Robert Shiller's institutional investor survey. The survey, conducted at

[7] We don't consider the II survey because it is not publicly available. And, we exclude the Michigan survey because of the short time period of its availability.

Figure 4 – Survey Data on Individual Investors' Expectations



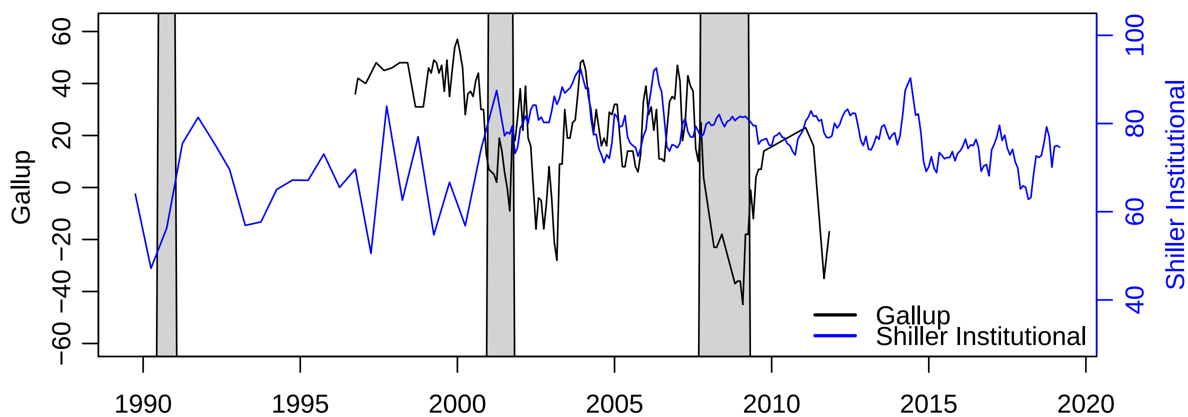
Notes: The figure plots the time series of investors' expectations about the stock market from the Gallup survey (solid black line), the Graham-Harvey survey of CFOs (red-dashed line), the American Association (AA) of Individual Investors Investor Sentiment Survey (blue-dotted line), and the Shiller Individual Investor Survey (green dotted-dashed line). The shaded areas denote the NBER-designated recession episodes.

six-month intervals over the period July 1989 to July 2001 and monthly thereafter, asks a sample of institutional investors (sampled from the investment managers section of the Money Market Directory of Pension Funds and Their Investment Managers) how much of a relative change they expect in the Dow Jones Industrial Index in the coming year. The U.S. Institutional One-Year Confidence Index is the percentage of institutional investors expecting an increase in the Dow in the coming year.

The survey reveals that the expectations of institutional investors differ markedly from those of individual investors. Specifically, the correlation between the U.S. Institutional One-Year Confidence Index and the Gallup survey is negative at −11.5%. The correlations between the Institutional Index and the GH and AA surveys are small at 8.3% and 18.8%, respectively, and are statistically indistinguishable from zero. The Shiller Ind is the only individual investor survey that has a statistically significant positive correlation of 52.6% with the Institutional Index. Moreover, unlike the individual investors' beliefs contained in surveys of individual investors, the institutional investors' beliefs are countercyclical, albeit somewhat mildly – the correlation between the Institutional index and the NBER recession dummy is 15.0%. In other words, institutional investors' expectations of the stock market tend to rise during bad times, consistent with the predictions of rational expectations based representative agent models. To further highlight the differences between the individual and institutional investors' beliefs, Figure 5 plots the time series of investors' expectations from the Gallup survey (black line) and the U.S. Institutional One-Year Confidence Index from Shiller's institutional investor survey (blue line).

Recall that our recovered beliefs about the stock market for the Type I investor suggest countercyclical expected market returns. This finding is in line with the reported beliefs in the above institutional investor survey.

Figure 5 – Individual Versus Institutional Investors' Expectations



Notes: The figure plots the time series of investors' expectations about the stock market from the Gallup survey (black line) and the U.S. Institutional One-Year Confidence Index from Shiller's Institutional Investor Survey (blue line). The shaded areas denote the NBER-designated recession episodes.

To summarize, the survey data point toward the existence of at least two types of investors with very different beliefs about the aggregate stock market. Recall that the beliefs we recovered from observed asset prices using the SEL methodology in Section IV also suggested the presence of two different cohorts of investors, differing from each other both in terms of their risk preferences as well as their beliefs about the future. The typical investor in large cap stocks resembles those in the Shiller institutional survey, in that they have countercyclical beliefs about the expected market return. To the contrary, the typical investor in small-growth companies is similar to the professional individual investors covered in the individual investor surveys, having pro cyclical beliefs about the stock market return. Thus, our findings offer a potential reconciliation of the seemingly contradictory evidence of procyclical expected market returns in survey data on investors' expectations versus the countercyclical expected returns implied by rational expectations representative agent models

VI The Data-generating Process Versus Investor Beliefs

Since beliefs vary substantially across investor types, a natural question is how beliefs compare to the "true" physical data-generating process. Section VI.1 shows that the SEL methodology can shed light on this question through an alternative, information-theoretic interpretation of the estimator. This interpretation reveals that the SEL objective function measures deviations of the recovered beliefs from non-parametrically estimated physical probabilities. Section VI.2 characterizes these deviations for the Type I and Type II investors.

VI.1 An Alternative Interpretation of the SEL estimator

Besides its relation to non-parametric maximum likelihood estimators, the SEL estimator also has an alternative information-theoretic interpretation (see, e.g., Kitamura and Stutzer (1997)). To see this, consider the type-l investor. Let $\mathcal{P}_t$ be the set of all conditional probability measures defined on $\mathbb{R}^q$, where q denotes the dimension of $(F_{t+1}^{(l)}, X_{t+1}, R_{t+1}^e)$. Put differently, q is the number of unique variables entering the SDF, the conditioning set, and the cross section of test assets used in the estimation. For any admissible SDF parameter $\gamma^{(l)} \in \Theta$, we define the set of conditional probability measures, absolutely continuous with respect to the (true) physical measure $\mathbb{P}_t^{phys}$, that satisfy the conditional Euler equations:

$$\mathcal{P}_t^{(l)}(\gamma^{(l)}) := \left\{ \mathbb{P}_t^{(l)} \in \mathcal{P}_t : \mathbb{E}^{\mathbb{P}_t^{(l)}} \left[M\left(F_{t+1}^{(l)}; \gamma^{(l)}\right) R_{i,t+1}^e \middle| X_t \right] = 0 \right\}, \forall t \in \{1, \ldots, T\}, i \in \{1, \ldots, I\}. \tag{14}$$

Therefore, $\mathcal{P}_t^{(l)}(\gamma^{(l)})$ is the set of all conditional probability measures, i.e. beliefs, that are consistent with the conditional Euler equations for this investor.

The SEL estimation can then be shown to select a probability measure $\widehat{\mathbb{P}_t^{(l)}(\gamma^{(l)})}$, for each t, such that:

$$\widehat{\mathbb{P}_t^{(l)}(\gamma^{(l)})} = \inf_{\mathbb{P}_t^{(l)} \in \mathcal{P}_t^{(l)}(\gamma^{(l)})} \text{KLIC}(\mathbb{P}_t^{phys}, \mathbb{P}_t^{(l)}) \equiv \inf_{\mathbb{P}_t^{(l)} \in \mathcal{P}_t^{(l)}(\gamma^{(l)})} \int \log \left(\frac{d\,\mathbb{P}_t^{phys}}{d\,\mathbb{P}_t^{(l)}} \right) d\,\mathbb{P}_t^{phys}$$
$$\text{s.t.} \quad \mathbb{E}^{\mathbb{P}_t^{(l)}} \left[M\left(F_{t+1}^{(l)}; \gamma^{(l)}\right) R_{t+1}^e \middle| X_t \right] = \mathbf{0}, \tag{15}$$

where $\text{KLIC}(\mathbb{P}_t^{phys}, \mathbb{P}_t^{(l)})$ is the Kullback-Leibler Information Criterion (KLIC) divergence (or, relative entropy) between the two measures $\mathbb{P}_t^{phys}$ and $\mathbb{P}_t^{(l)}$ (see White (1982)).

The KLIC divergence is non-negative, and is exactly equal to zero if and only if $\mathbb{P}_t^{(l)} = \mathbb{P}_t^{phys}$ almost surely, that is, if the type-l investors' beliefs (that satisfy her conditional Euler restrictions for the test assets) coincide with the data-generating process. Thus, the SEL approach searches for an estimate of $\mathbb{P}_t^{(l)}$ that makes the estimated beliefs as close as possible – in the information-theoretic sense – to the physical measure $\mathbb{P}_t^{phys}$, subject to the constraint that the estimated beliefs satisfy the pricing restrictions given by the conditional Euler equations.

To make (15) operational, we need an estimate for the physical measure. A natural choice is the kernel density $\mathbb{P}_t^{phys} = \{\omega_{t,j}\}_{j=1}^T$, a widely used nonparametric estimate of conditional distributions. This estimate maximizes the log-likelihood in Equation (5) when the conditional Euler equation restrictions are *not* imposed. In other words, the maximum likelihood estimate of beliefs in the absence of any asset pricing restrictions simply equals the kernel density weights. With the kernel density estimate of the physical probabilities, Equation (15) becomes

$$\widehat{\mathbb{P}_t^{(l)}(\gamma^{(l)})} = \min_{\left\{p_{t,j}^{(l)}\right\}_{j=1}^T} \sum_{j=1}^T \log\left(\frac{\omega_{t,j}}{p_{t,j}^{(l)}}\right)\omega_{t,j}, \text{ s.t. } \sum_{j=1}^T p_{t,j}^{(l)} M\left(f_j^{(l)};\gamma^{(l)}\right) r_{i,j}^e = 0 \qquad (16)$$

A comparison of Equations (5) and (16) shows that the two have identical solutions.

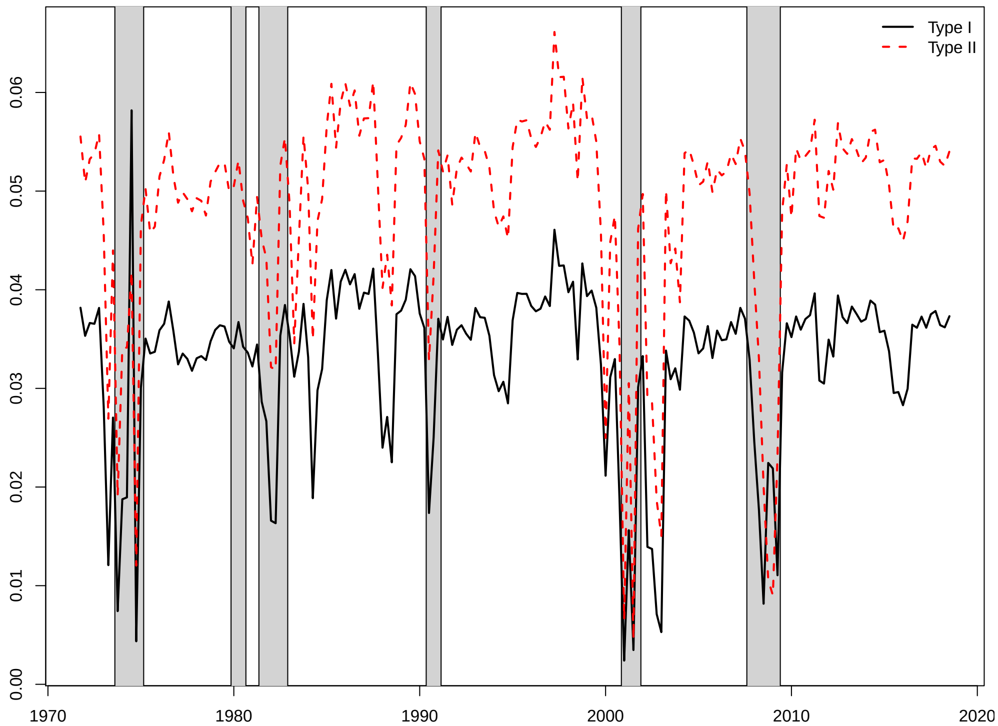
VI.2 Empirical Results

Figure 6 presents the time series of the KLIC divergence for the Type I (black solid line) and Type II (red dashed line) investors' beliefs. Recall that this divergence, for each state t, is given by the objective function in Equation (16), and quantifies the discrepancy between the recovered subjective beliefs of the type-l investor and the physical DGP.

Figure 6 shows that the beliefs of Type I investors, who invest primarily in large-cap stocks, tend to be closer to the physical data-generating probabilities than the beliefs of Type II investors, who heavily allocate to small-growth equities. This, however, does not imply that Type I investors are necessarily more rational than Type II's, since the marginal investor, who sets prices, may not be rational. At the same time, to the extent that investors are price takers, it may not be rational to deviate from the beliefs that set prices even if they know these beliefs are irrational. Given these issues, we refrain from making strong conclusions about rationality.

Another salient feature from Figure 6 is that the beliefs divergence from the physical probabilities tends to be smaller in recessions for both investor types. The correlation between

Figure 6 – KLIC Divergence

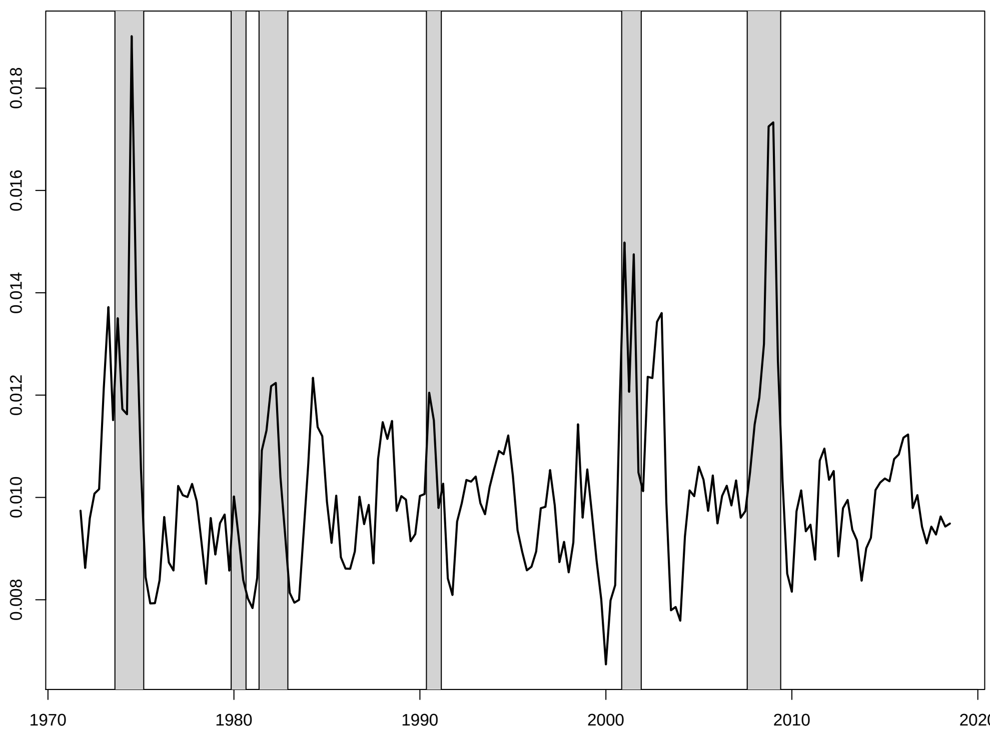


Notes: The figure plots the time series of KLIC divergence for the two investor types from the physical data generating probabilities. The black line is the divergence for Type-I investors, who invest in large-cap stocks. The red dashed line shows the divergence for Type-II investors, who invest in small-growth stocks. The physical distribution is computed using a non-parametric kernel density estimator. The investors' beliefs are obtained using the estimated SEL distributions. The pricing kernel is exponentially affine in the large cap portfolio return for the Type I investor and in the small-growth portfolio return for the Type II investor. The test assets consist of the excess returns on the large cap and small growth portfolios. The conditioning set consists of the averages of the last four quarter's large cap and small-growth portfolio returns. The sample is quarterly covering the period 1972:Q1-2018:Q4.

a recession dummy and the KLIC divergence is -47.8% for Type-I investors and -54.0% for Type-II investors. At the same time, Figure 7, that plots the KLIC divergence between the beliefs of the Type I and II investors, $\left\{\sum_{j=1}^{T} \log\left(\frac{p_{t,j}^{(I)}}{p_{t,j}^{(II)}}\right) p_{t,j}^{(I)}, \quad t = 1, 2, ..., T\right\}$, shows that the two investor types *disagree* more with each other during economic downturns. Specifically, the correlation between the KLIC divergence and the recession dummy is 52.6%.

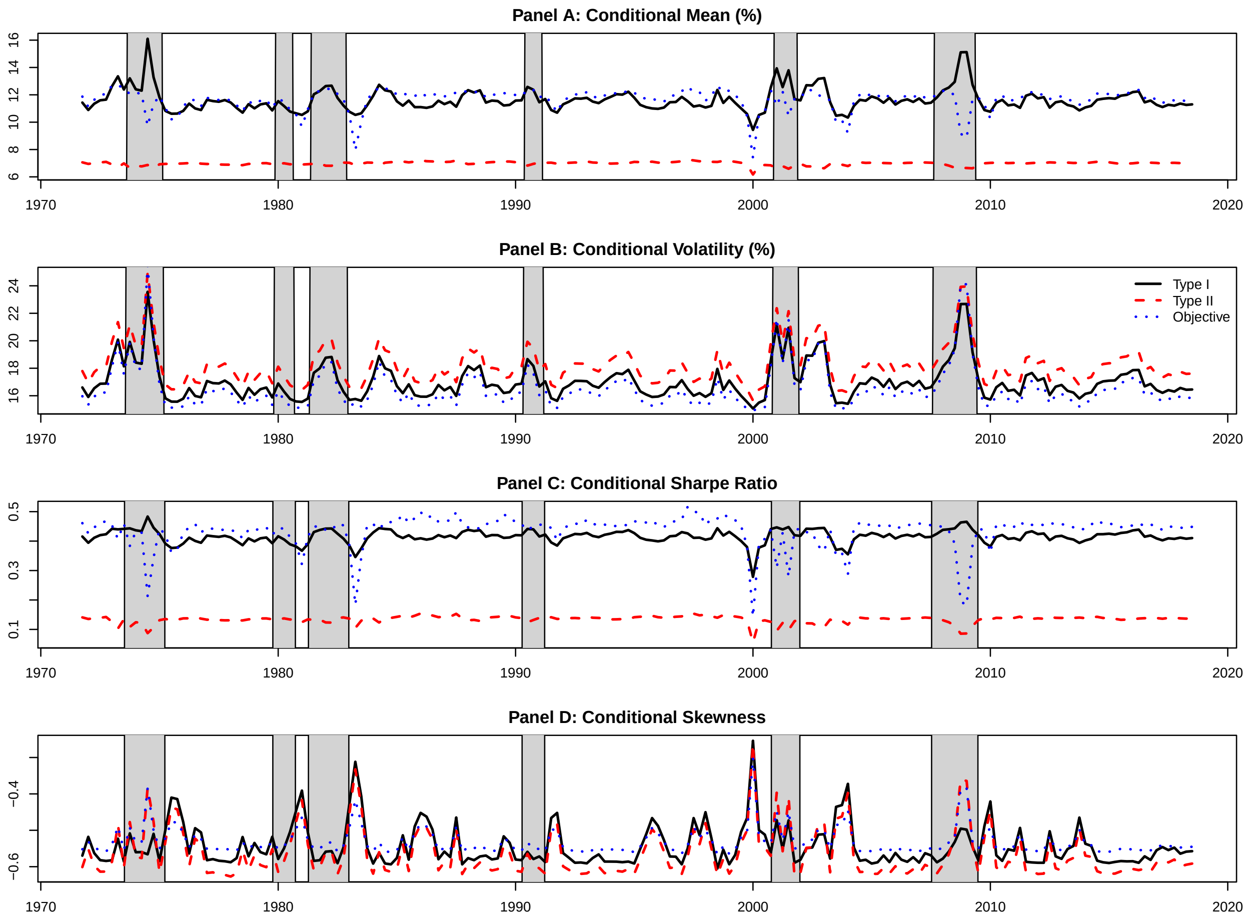
To appreciate the source of this apparent paradox, Figures 8 and 9 plot the time series of the expected returns (Panel A), volatility (Panel B), Sharpe ratio (Panel C), and skewness (Panel D) of the market portfolio and the small-growth portfolio, respectively, implied by the recovered beliefs of the Type I investor (black solid line), Type II investor (red dashed line), as well as the physical probabilities (blue dotted line). In recessions, the Type I investor's beliefs tend to diverge from the physical market return distribution, but are closer to the physical distribution of the small-growth portfolio, compared to expansions. The opposite happens for the Type II investor, namely her beliefs get closer to the physical distribution of the market return but farther from the physical distribution of the small-growth portfolio return during recessionary episodes. Thus, while both investors' beliefs are on the whole closer to the joint distribution of the physical return-generating process during recessions, the movement is on two different dimensions (the market versus the small-growth portfolio). In each dimension, their beliefs in fact move away from each other, explaining the growing disagreement between the two investors in economic downturns.

Figure 7 – Beliefs Disagreement: KLIC Divergence Between Type I and II Investors



Notes: The figure plots the time series of KLIC divergence between Type-I investors, who invest in large-cap stocks, and Type-II investors, who invest in small-growth stocks. The investors' beliefs are obtained using the estimated SEL distributions. The pricing kernel is exponentially affine in the large cap portfolio return for the Type I investor and in the small-growth portfolio return for the Type II investor. The test assets consist of the excess returns on the large cap and small growth portfolios. The conditioning set consists of the averages of the last four quarter's large cap and small-growth portfolio returns. The sample is quarterly covering the period 1972:Q1-2018:Q4.

Figure 8 – Objective Versus Recovered Beliefs About the Market Portfolio

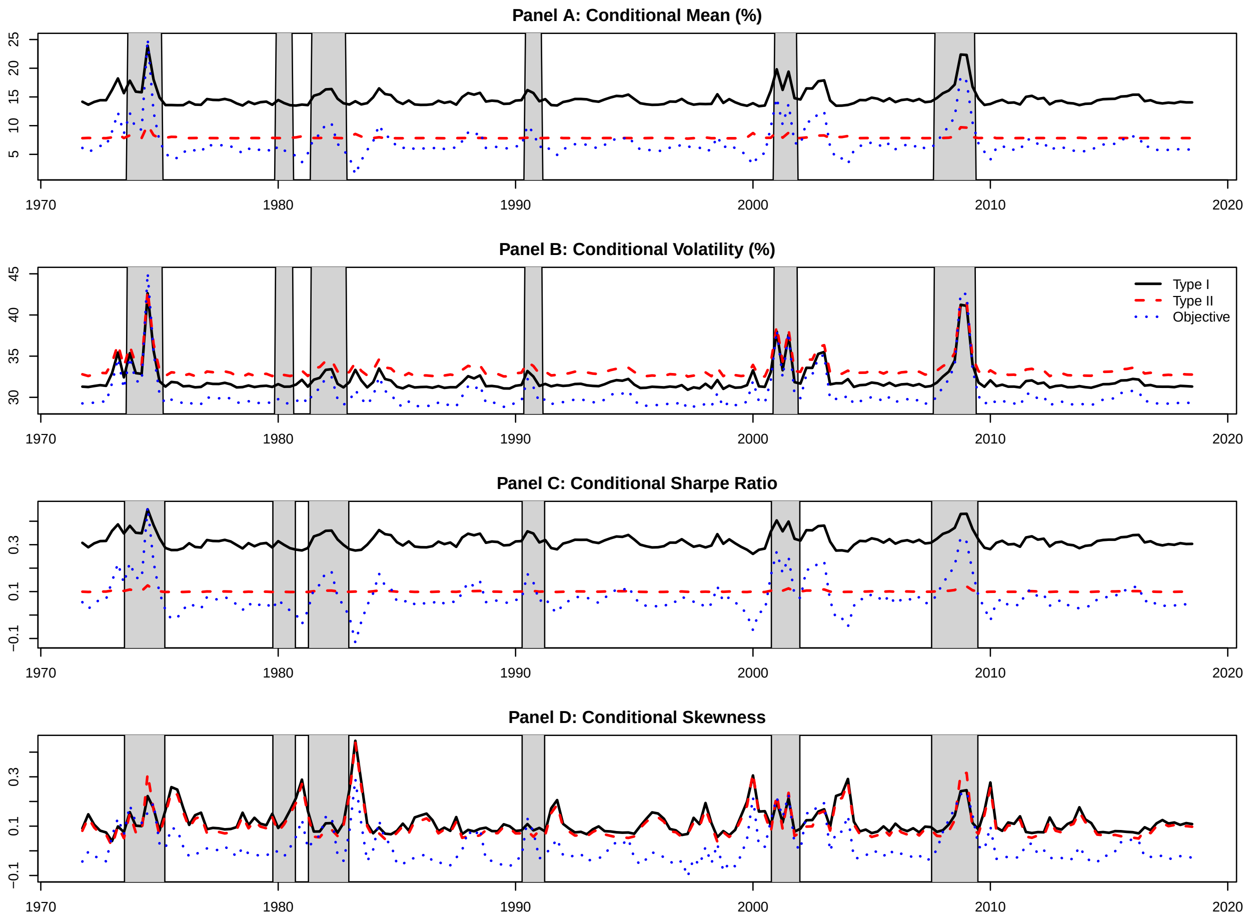


Notes: The figure plots the time series of the conditional mean (Panel A), volatility (Panel B), Sharpe ratio (Panel C), and skewness (Panel D) of the market return, as perceived by the Type I investor (solid black like) and Type II investor (red-dashed line), and the physical return distribution (blue dotted line). Types are as described in Table 1. Shaded areas denote NBER designated recession periods. The physical return moments are computed from a non-parametric kernel density estimator of the joint distribution of returns. The investors' conditional moments are obtained using the estimated SEL distributions. The pricing kernel is exponentially affine in the large cap portfolio return for the Type I investor and in the small-growth portfolio return for the Type II investor. The test assets consist of the excess returns on the large cap and small growth portfolios. The conditioning set consists of the averages of the last four quarter's large cap and small-growth portfolio returns. The sample is quarterly covering the period 1972:Q1-2018:Q4.

VII Robustness

Thus far we have focused on two extreme investor types, namely those that invest all of their wealth in large market cap stocks and those that invest only in small growth stocks. In this section, we show robustness of our results to changing the investors' wealth portfolios to include other asset mixes. We also establish the robustness of our main findings to the

Figure 9 – Objective Versus Recovered Beliefs About the Small-Growth Portfolio



Notes: The figure plots the time series of the conditional mean (Panel A), volatility (Panel B), Sharpe ratio (Panel C), and skewness (Panel D) of the small-growth portfolio return, as perceived by the Type I investor (solid black like) and Type II investor (red-dashed line), and the physical return distribution (blue dotted line). Types are as described in Table 1. Shaded areas denote NBER designated recession periods. The physical return moments are computed from a non-parametric kernel density estimator of the joint distribution of returns. The investors' conditional moments are obtained using the estimated SEL distributions. The pricing kernel is exponentially affine in the large cap portfolio return for the Type I investor and in the small-growth portfolio return for the Type II investor. The test assets consist of the excess returns on the large cap and small growth portfolios. The conditioning set consists of the averages of the last four quarter's large cap and small-growth portfolio returns. The sample is quarterly covering the period 1972:Q1-2018:Q4.

choice of sample period, data frequency, and test assets.

Table 2 presents evidence that our results are robust to the choice of sample period as well as data frequency. Panel A shows results obtained using a monthly (instead of quarterly) frequency over the same sample period (1972–2018) as the baseline results in Table 1. The results are very similar to Table 1. When the large cap and small-growth portfolios are constructed from the 25 FF (Row 1) and 6 FF (Row 2) portfolios using the previous month's returns on these two portfolios as conditioning variables, the point estimates of the SDF parameter for the Type I investor in Column 2 are 2.4 and 2.7, respectively. Very similar estimates of 2.5 and 2.8, respectively, are obtained in Column 3 when the averages of the last twelve months' returns on the large cap and small-growth portfolios are used in the conditioning set.

Table 2 – Estimates of Preference Parameters: Robustness

	$\gamma^{(i)}$		LR Test ($H_0 : \gamma^{(I)} = \gamma^{II}$)	
	Cond. Set 1	Cond. Set 2	Cond. Set 1	Cond. Set 2
	Panel A: Monthly 1972:01–2018:12			
Type-I: B^{FF25}, SG^{FF25}	2.4 (0.99)	2.5 (1.00)	4.71 [.030]	5.14 [.023]
Type-I: B^{FF6}, SG^{FF6}	2.7 (0.99)	2.8 (1.00)	3.54 [.060]	3.42 [.064]
Type-II: B^{FF25}, SG^{FF25}	0.3 (0.55)	0.3 (0.56)	-	-
Type-II: B^{FF6}, SG^{FF6}	0.9 (0.63)	1.0 (0.64)	-	-
	Panel B: Quarterly 1947:01–2018:12			
Type-I: B^{FF25}, SG^{FF25}	2.5 (0.78)	2.8 (0.80)	8.17 [.004]	9.79 [.002]
Type-I: B^{FF6}, SG^{FF6}	2.7 (0.78)	3.0 (0.81)	5.41 [.020]	6.51 [.011]
Type-II: B^{FF25}, SG^{FF25}	0.4 (0.40)	0.5 (0.41)	-	-
Type-II: B^{FF6}, SG^{FF6}	1.0 (0.49)	1.1 (0.50)	-	-

The table reports the point estimates of the SDF parameter $\gamma^{(l)}$, along with the asymptotic standard errors in parentheses below, for the Type-I investor (Rows 1 and 2 of each panel) and Type-II investor (Rows 3 and 4 of each panel). The estimates are obtained using the SEL method, with the SDF exponentially affine in the large cap portfolio return for Type-I investors and in the small-growth portfolio return for Type-II investors. The conditioning set consists of the last quarter's large cap and small-growth returns (Cond. Set 1; Column 2) and the average of the last four quarter's large cap and small-growth returns (Cond. Set 2; Column 3). Columns 4 and 5 report likelihood ratio tests, with the associated p-values in square brackets below, of the null hypothesis that $\gamma^{(l)}$ is equal across the two investor types. The test statistic has a $\chi^2_{(1)}$-distribution under the null. The sample is monthly, covering the period $1972:01 - 2018:12$ (Panel A) and quarterly, covering the entire post war period $1947Q1 - 2018Q4$ (Panel B).

For the Type II investor, using the 25 FF portfolios for the two test assets (Row 3), the point estimate of the SDF parameter is 0.3 for either choice of conditioning information. Similar to Table 1, this is at least eight times smaller than the corresponding estimates obtained for the Type I investor in Row 1. Also similar to Table 1, when the test assets are constructed from the 6 FF portfolios (Row 4 versus Row 2), the point estimate of this parameter is about three times smaller than those obtained for the Type I investor. Finally, the results remain largely similar over the longer post war quarterly sample $1947Q1-2018Q4$, and are presented in Panel B. The results for the likelihood ratio tests in both panels are similar to, albeit even stronger than, those reported in Table 1 – the null hypothesis of equal SDF parameters for the two investor types is rejected at the 5% significance level in 6 out of 8 cases (two data frequencies, two choices of the conditioning set, and two approaches to constructing the large cap and small growth equity portfolios) and at the 10% level in all 8 cases.

We next show that the procyclical beliefs about the expected market return is a unique features of investors who allocate heavily to small-growth types financial securities. Table 3, Panel A presents the recovered risk preferences and beliefs for four investor types whose portfolios correspond to the intersection of the smallest and largest quintiles of the size and BE/ME sorted portfolios: the small-growth portfolio (SG in Row 1), the small-value portfolio (SV in Row 2), the big-growth portfolio (BG in Row 3), and the big-value portfolio (BV in Row 4). The results suggest that the small-growth investor type is unique, both in terms of the very low estimate of their risk aversion parameter (0.3 in Row 1, Column 2) and their procyclical beliefs about the expected market return (-25.9% correlation with the recession dummy in Row 1, Column 3) and its Sharpe ratio (-29.0% correlation with the recession dummy in Row 1, Column 5). Rows 2–4 show that the other three investor types all have substantially higher risk aversions (varying from 2.0–2.3, at least seven times larger than the 0.3 estimate obtained for the SG investor) and countercyclical beliefs about the expected market return (the correlations with the recession dummy vary from 26.3% to 56.9%, compared to -25.9% obtained for the SG investor) and its Sharpe ratio (the correlations with the recession dummy vary from 20.8% to 58.4% compared to -29.0%, obtained for the SG investor). Figure 10 in Appendix B plots the time series of the expected market return (Panel A) and the Sharpe ratio of the market return (Panel B) for the four investor types.

We next show that the low value of the estimated risk aversion parameter and the procyclical beliefs about the expected market return for the Type II (SG) investor are not driven by the extreme assumption that such investors invest solely in small-growth type assets. We do this by allowing such investors to allocate between small-growth equities and risk free short term Treasury bonds (TBills), such that their chosen portfolio is $R_W = xR_f + (1-x)R_{SG}$. Table 3, Panel B presents the recovered SDF parameter and beliefs for different choices of x. Not surprisingly, the SDF parameter estimate increases with x, consistent with the intuition that more risk averse investors would allocate more heavily to safe government bonds over equities. However, Panel B, Row 4 shows that the estimated risk aversion parameter

Table 3 – Beliefs About the Stock Market and Risk Preferences Across Investors with Different Optimal Portfolios

	$\gamma^{(i)}$	$\rho\left(E^{\mathbb{P}^{(i)}}(R_m), I_{rec}\right)$ %	$\rho\left(Vol^{\mathbb{P}^{(i)}}(R_m), I_{rec}\right)$ %	$\rho\left(SR^{\mathbb{P}^{(i)}}(R_m), I_{rec}\right)$ %
		Panel A: $R_W = R_n$		
$n = SG$	0.3	-25.9	53.0	-29.0
$n = SV$	2.3	56.0	59.8	56.3
$n = BG$	2.1	56.9	56.8	58.4
$n = BV$	2.0	26.3	40.2	20.8
		Panel B: $R_W = xR_f + (1-x)R_{SG}$		
$x = 0.0$	0.30	-25.9	53.0	-29.0
$x = 0.2$	0.35	-30.9	53.0	-29.8
$x = 0.4$	0.45	-32.8	53.0	-29.8
$x = 0.6$	0.60	-38.9	53.0	-31.0
$x = 0.9$	0.95	-54.0	52.8	-37.3
		Panel C: $R_W = 0.2R_f + xR_{SG} + (1-0.2-x)R_m$		
$x = 0.8$	0.35	-30.9	53.0	-29.8
$x = 0.6$	0.60	15.3	53.2	-14.4
$x = 0.4$	1.00	40.3	53.5	11.0
$x = 0.2$	1.70	48.7	54.0	34.2
$x = 0.0$	2.90	51.9	54.6	44.6

The table reports the point estimates of the SDF parameter $\gamma^{(l)}$ (Column 2), and correlations between a NBER recession dummy, I_{rec}, and the recovered beliefs about the market's expected return (Column 3), volatility (Column 4), and Sharpe ratio (Column 5), for different investor types. Panel A presents results for investors who invest in one of the corner portfolios from the 25 Fama-French portfolios: the small growth portfolio (SG, Row 1), the small value portfolio (SV, Row 2), the big growth portfolio (BG, Row 3), or the big value portfolio (BV, Row 4). Panel B presents results for investors who allocate between small-growth equities and short term Treasury bonds. Finally, Panel C presents results for investors who allocate between large cap equities, small-growth equities, and short term Treasury bonds. The SDF parameter estimates and beliefs are obtained using the SEL approach. For each investor type, the pricing kernel is exponentially affine in their chosen portfolio. The test assets consist of the excess return on the market portfolio and the investor's chosen portfolio. The conditioning set consists of the averages of the last four quarter's large cap and small-growth portfolio returns. The sample is quarterly covering the period 1972:Q1-2018:Q4.

remains less than half of those obtained for the SV, BG, and BV investors in Panel A, even for investors who allocate 90% to TBills and the remaining 10% to small-growth stocks (0.95 versus 2.0–2.3). Moreover, SG investors have procyclical beliefs about the expected market return (Column 3) and its Sharpe ratio (Column 5) regardless of their allocation to TBills. Figure 11 in Appendix B plots the time series of the expected market return (Panel A) and the Sharpe ratio of the market return (Panel B) for different values of x.

Table 3, Panel C goes a step further and examines the risk preferences and beliefs of investors who allocate between three asset classes – large cap equities, small-growth equities, and short term TBills. We assume that these investors allocate 20% to TBills and split the remainder between the two risky equity portfolios, i.e. their chosen portfolio is $R_W = 0.2R_f + xR_{SG} + (1-0.2-x)R_m$. Panel C shows that the procyclical beliefs about the expected market return obtain for investors with at least 70% allocation to small-growth equities. For the expected Sharpe ratio of the market, procyclical beliefs are obtained for investors with at least 60% allocation to small-growth equities. For investors with smaller allocations to small-growth stocks, the recovered beliefs about the expected market return and its Sharpe ratio are countercyclical. Not surprisingly, for investors with 20% allocation to TBills, the estimated risk aversion parameter rises from 0.35 from those with 80% allocation to small-growth equities and 0% to large cap stocks to almost nine times higher at 2.90 for those with 0% allocation to small-growth equities and 80% to large cap stocks. Figure 12 in Appendix B plots the time series of the expected market return (Panel A) and the Sharpe ratio of the market return (Panel B) for different values of x.

Overall, the results that our key findings of procyclical beliefs about the expected market return for investors allocating heavily to small-growth type assets and countercyclical beliefs about the same for investors primarily investing in stable large cap equities, and the former investors being substantially less risk averse compared to the latter, are quite robust.

VIII Conclusion and Extensions

We propose a novel methodology to separately identify the risk preferences and beliefs about the future of investors in different segments of the financial market. Our approach is non-parametric, not requiring us to take a stance on either the origins or forms of beliefs heterogeneity across different market participants, or investor rationality or lack thereof. Instead, it relies on the insights that investors with different beliefs and/or risk preferences optimally choose different portfolio holdings and that asset prices reflect the beliefs and preferences of the investors who trade in them. Based on these insights, we use observed asset prices to recover the beliefs and preferences of these heterogeneous investors.

Our results suggest the presence of at least two types of investors, differing from each other both in terms of risk preferences and beliefs. Investors like the ones who primarily invest in large cap equities are more risk averse and believe that the expected stock market return is strongly countercyclical. On the other hand, investors of the type that invest heavily in small-growth equities (often regarded as the public market counterpart of venture capital backed start-up companies) are substantially less risk averse, with their beliefs implying strong procyclicality in the expected stock market return. Our findings offer a potential reconciliation of the seemingly contradictory evidence of procyclical beliefs about the market return in survey data on investors' expectations about the stock market versus the countercyclical expected market returns implied by rational expectations representative agent models.

Finally, in the current work, we apply our methodology to two segments of the US public equity market. However, our methodology is considerably general and may be applied to recover the beliefs and risk preferences of investors whose portfolio holdings include more exotic asset classes, such as currencies, commodities, private equity, venture capital, and derivative securities. These are left for future research.

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A Appendix: SEL Estimator Properties

This appendix describes the asymptotic as well as the finite-sample properties of the SDF parameter estimates and beliefs distributions recovered using the SEL methodology. We present these properties separately for two cases: (a) the conditional Euler equation is correctly specified, i.e. Equation (2) holds with $\mathcal{P}^{(l)} = \mathcal{P}_0$, the latter being the objective distribution of the data; and (b) the Euler equation is mis-specified, such that Equation (2) holds with $\mathcal{P}^{(l)} = \mathcal{P}^{*(l)} \neq \mathcal{P}_0$.

Consider first the case of the correctly specified model. In this case, Kitamura, Tripathi, and Ahn (2004) show that the SEL approach delivers a consistent and asymptotically efficient estimator of $\gamma^{(l)}$, i.e. the estimator achieves the semi-parametric efficiency bound in Chamberlain (1987) for conditional moment restriction models. Thus, we have

$$T^{1/2}\left(\widehat{\gamma^{(l)}}^{SEL} - \gamma_0^{(l)}\right) \xrightarrow{d} N\left(0, I^{-1}\left(\gamma_0^{(l)}\right)\right), \tag{17}$$

where $\gamma_0^{(l)}$ is the true value of the parameter, and

$$I^{-1}\left(\gamma_0^{(l)}\right) = \left\{E\left[D'(x_t)V^{-1}(x_t)D'(x_t)\right]\right\}^{-1}. \tag{18}$$

In the above equation, $D(x_t) \equiv E\left[\frac{\partial}{\partial\left(\gamma^{(l)}\right)'}\left(M\left(F_{t+1}^{(l)};\gamma_0^{(l)}\right)R_{i,t+1}^e\right)|X_t = x_t; \gamma^{(l)} = \gamma_0^{(l)}\right]$ denotes the derivative of the conditional Euler equations with respect to the SDF parameters, and $V(x_t) \equiv E\left[\left(M\left(F_{t+1}^{(l)};\gamma_0^{(l)}\right)R_{i,t+1}^e\right)\left(M\left(F_{t+1}^{(l)};\gamma_0^{(l)}\right)R_{i,t+1}^e\right)'|X_t = x_t; \gamma^{(l)} = \gamma_0^{(l)}\right]$ denotes the covariance matrix of the moment restrictions. In our empirical implementation, we compute non-parametric kernel estimates for the conditional moments underlying $D(x_t)$ and $V(x_t)$ and, thereby, obtain estimates of the asymptotic variance of the estimated SDF parameters using Equation (18).

Since the SEL approach is likelihood-based, we can perform likelihood-ratio tests to conduct inference on the SDF parameter $\gamma^{(l)}$. Suppose that we want to test the parametric restriction $H_0 : R\left(\gamma_0^{(l)}\right) = 0$, against $H_1 : R\left(\gamma_0^{(l)}\right) \neq 0$, where $R\left(\gamma_0^{(l)}\right)$ is an $(r \times 1)$ vector and $r \leq d$ and d denotes the dimension of $\gamma^{(l)}$. Define:

$$\widehat{\gamma^{(l)}}^{R} = \underset{\gamma^{(l)}\in\Theta}{\operatorname{argmax}}\, SEL\left(\gamma^{(l)}\right), \quad s.t. \quad R\left(\gamma^{(l)}\right) = 0\,. \tag{19}$$

Kitamura, Tripathi, and Ahn (2004) show that the above test can be performed using a SEL version of the likelihood-ratio test statistic. Specifically, under the null hypothesis,

$$LR_T = 2\left\{SEL\left(\widehat{\gamma^{(l)}}^{SEL}\right) - SEL\left(\widehat{\gamma^{(l)}}^{R}\right)\right\} \xrightarrow{d} \chi_{(r)}. \tag{20}$$

The advantage of the likelihood ratio test in conducting inference on the SDF parameter is that it, unlike the t-test, does not require the estimation of the asymptotic covariance matrix, $I\left(\gamma_0^{(l)}\right)$ in Equation (17), obtaining good estimates of which may be difficult in practice.

Also, in this case, Brown and Newey (1998) and Kitamura, Tripathi, and Ahn (2004) show that the SEL approach provides a consistent and semi-parametrically efficient estimator of the beliefs (or, equivalently, the true underlying data generating process) $\mathcal{P}_0$. Consider, for example, the estimation of the expected market return at time (or, state) t. A standard non-parametric kernel density estimator yields:

$$E^{kernel}\left[\widehat{R_{m,t+1}|}X_t = x_j\right] = \sum_{k=1}^{T} \underbrace{\left\{\frac{\mathcal{K}\left(\frac{x_j - x_k}{b_T}\right)}{\sum_{k=1}^{T}\mathcal{K}\left(\frac{x_j - x_k}{b_T}\right)}\right\}}_{\omega_{j,k}} r_{m,k}. \tag{21}$$

The estimator of the conditional expectation formed by investor type l, on the other hand, is given by

$$E^{(l)}\left[\widehat{R_{m,t+1}|}X_t = x_j\right] = \sum_{k=1}^{T} \omega_{j,k}\left[\frac{1}{1 + M\left(F_k^{(l)}; \widehat{\gamma^{(l)}}^{SEL}\right) \cdot \lambda_j(\widehat{\gamma^{(l)}}^{SEL})\, r_{i,k}^{e}}\right] r_{m,k}. \tag{22}$$

Compared to Equation (21), the extra weighting factor (in square brackets) provides an efficiency gain, thereby yielding a semi-parametrically efficient estimator of the true conditional distribution. The intuition for the above result lies in the SEL estimator, unlike the kernel density estimator, taking advantage of the information contained in the moment restrictions (see also Brown and Newey (2002)). The superior numerical properties of an SEL-type estimator of the conditional distribution function are also demonstrated via simulations in Hall, Wolff, and Yao (1999).

Consider next the scenario where the conditional Euler equation is misspecified. This could happen, for instance, if the SDF summarizing investor type-l's preferences is correctly specified but these investors have distorted beliefs. In this case, Kitamura (2003) shows that, under certain regularity conditions, the SEL estimator of the parameter $\gamma^{(l)}$ converges in probability to a pseudo-true value $\gamma^{(l)*}$. He also shows that, under additional regularity conditions, the estimator has the following limiting Gaussian distribution:

$$T^{1/2}\left(\widehat{\gamma^{(l)}}^{SEL}-\gamma^{(l)*}\right)\xrightarrow{d}N\left(0,\Omega^{-1}\left(\gamma^{(l)*}\right)\right), \tag{23}$$

where

$$\Omega^{-1}\left(\gamma^{(l)*}\right)=\left\{E\left[D'(x_t)V^{-1}(x_t)D'(x_t)\right]\right\}^{-1}, \tag{24}$$

and

$$D(x_t)\equiv\frac{\partial}{\partial\left(\gamma^{(l)}\right)'}E\left[\frac{M\left(F_{t+1}^{(l)};\gamma^{(l)}\right)R_{i,t+1}^{e}}{1+\lambda_t'(\gamma^{(l)})\left(M\left(F_{t+1}^{(l)};\gamma^{(l)}\right)R_{i,t+1}^{e}\right)}|X_t=x_t;\gamma^{(l)}=\gamma^{(l)*}\right],$$

$$S(x_t)\equiv\frac{M\left(F_{t+1}^{(l)};\gamma^{(l)}\right)R_{i,t+1}^{e}}{1+\lambda_t'(\gamma^{(l)})\left(M\left(F_{t+1}^{(l)};\gamma^{(l)}\right)R_{i,t+1}^{e}\right)},$$

$$V(x_t)\equiv E\left[S(x_t)S(x_t)'|X_t=x_t;\gamma^{(l)}=\gamma^{(l)*}\right].$$

This result offers a generalization of those in Kitamura, Tripathi, and Ahn (2004), who consider correctly specified conditional moment restriction models. Note that, when the moment restrictions are correctly specified, the pseudo-true parameter value coincides with the true value of the parameter, i.e. $\gamma^{(l)*}=\gamma_0^{(l)}$, and the Lagrange multipliers associated with the conditional moment restrictions, $\lambda_t(\gamma_0^{(l)})$, converge to zero for all values of the conditioning set (or, state) as the sample size increases. Thus, the limiting distribution of the SEL estimator in the presence of model misspecification in Equation (24) coincides with that obtained under correct model specification in Equation (18).

Moreover, in this case of a misspecified model, the SEL estimator of the beliefs provides a consistent estimator for the pseudo-true conditional measure – the measure that has the minimum distance, in terms of the conditional Kullback-Leibler Information Criterion (KLIC) Divergence, from the true conditional measure:

$$\mathcal{P}^{(l)}(\widehat{\widehat{\gamma^{(l)}}^{SEL}})=\inf_{\gamma^{(l)}\in\Theta,\mathcal{P}\in\mathbb{P}_t(\gamma^{(l)})}\ \text{KLIC}(\mathcal{P}_0,\mathcal{P})\quad\equiv\quad\inf_{\gamma^{(i)}\in\Theta,\mathcal{P}\in\mathbb{P}(\gamma^{(i)})}\int\log\left(\frac{\mathrm{d}\,\mathcal{P}_0^{(\cdot|X_t)}}{\mathrm{d}\,\mathcal{P}^{(\cdot|X_t)}}\right)\mathrm{d}\,\mathcal{P}_0^{(\cdot|X_t)}$$
$$\text{s.t.}\quad\mathbb{E}^{\mathcal{P}}\left[M\left(F_{t+1}^{(l)};\gamma^{(l)}\right)R_{i,t+1}^{e}\middle|X_t\right]=\mathbf{0}, \tag{25}$$

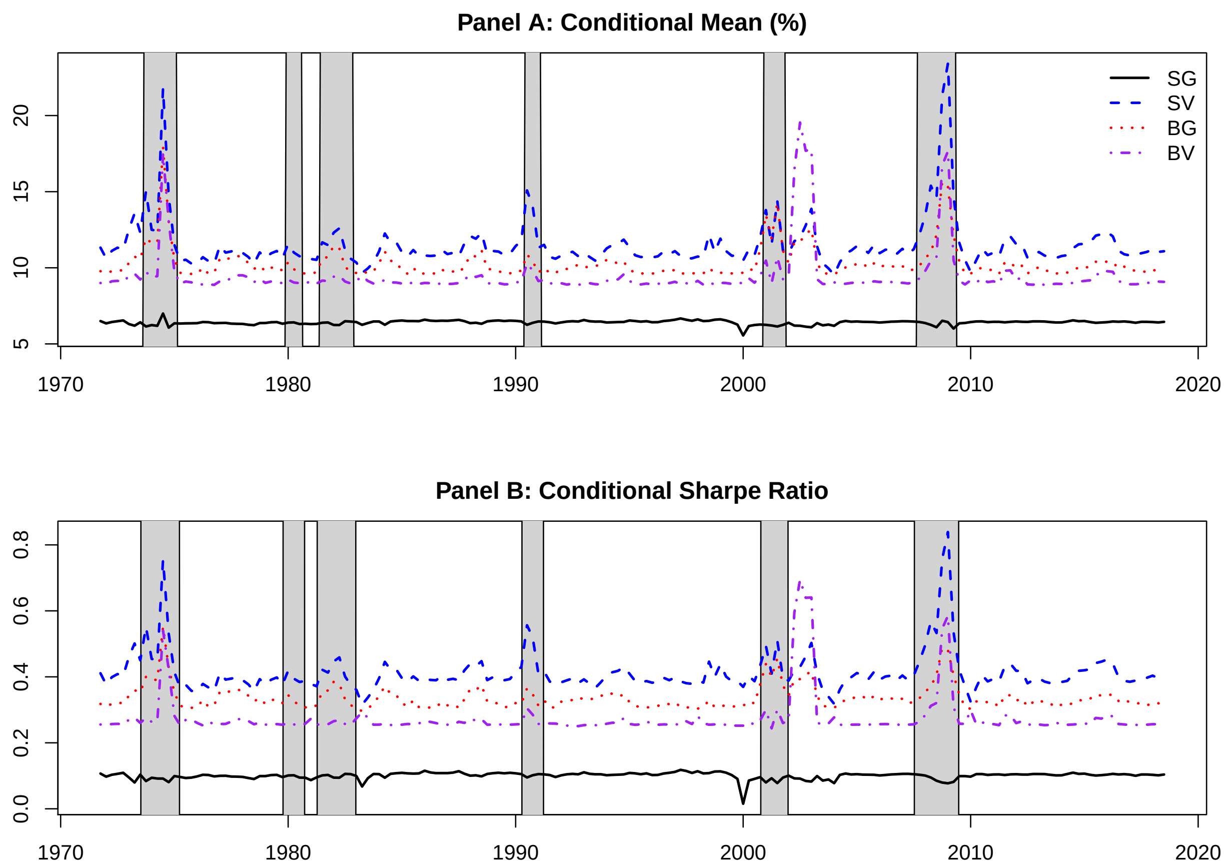
where the superscript $\mathcal{P}_0^{(\cdot|X_t)}$ is added to emphasize the conditional nature of the probability measures; $\text{KLIC}(\mathcal{P}_0,\mathcal{P})$ is the KLIC divergence (or, relative entropy) between the two measures $\mathcal{P}_0$ and $\mathcal{P}$ (see White (1982)); and $\mathbb{P}_t(\gamma^{(l)})$, $\gamma^{(l)}\in\Theta$, is the set of all conditional probability measures, absolutely continuous with respect to $\mathcal{P}_0$, that are consistent with the conditional Euler equation restrictions, for a given value of the SDF parameter:

$$\mathbb{P}_t(\gamma^{(l)}) := \left\{\mathcal{P} \in \mathbb{P} \,:\, \mathbb{E}^{\mathcal{P}}\left[M\left(F_{t+1}^{(l)};\gamma^{(l)}\right) R_{i,t+1}^{e} \middle| X_t\right] = \mathbf{0}\right\}, \forall t \in \{1,\ldots,T\}. \tag{26}$$

In this case, the role of the extra multiplicative factor in the SEL-estimated distribution (see Equation (22)) is to adjust the standard kernel density estimator so that it converges to the pseudo-true measure. This property parallels that of parametric maximum likelihood estimators for misspecified models (see, e.g., White (1982), Vuong (1989)).

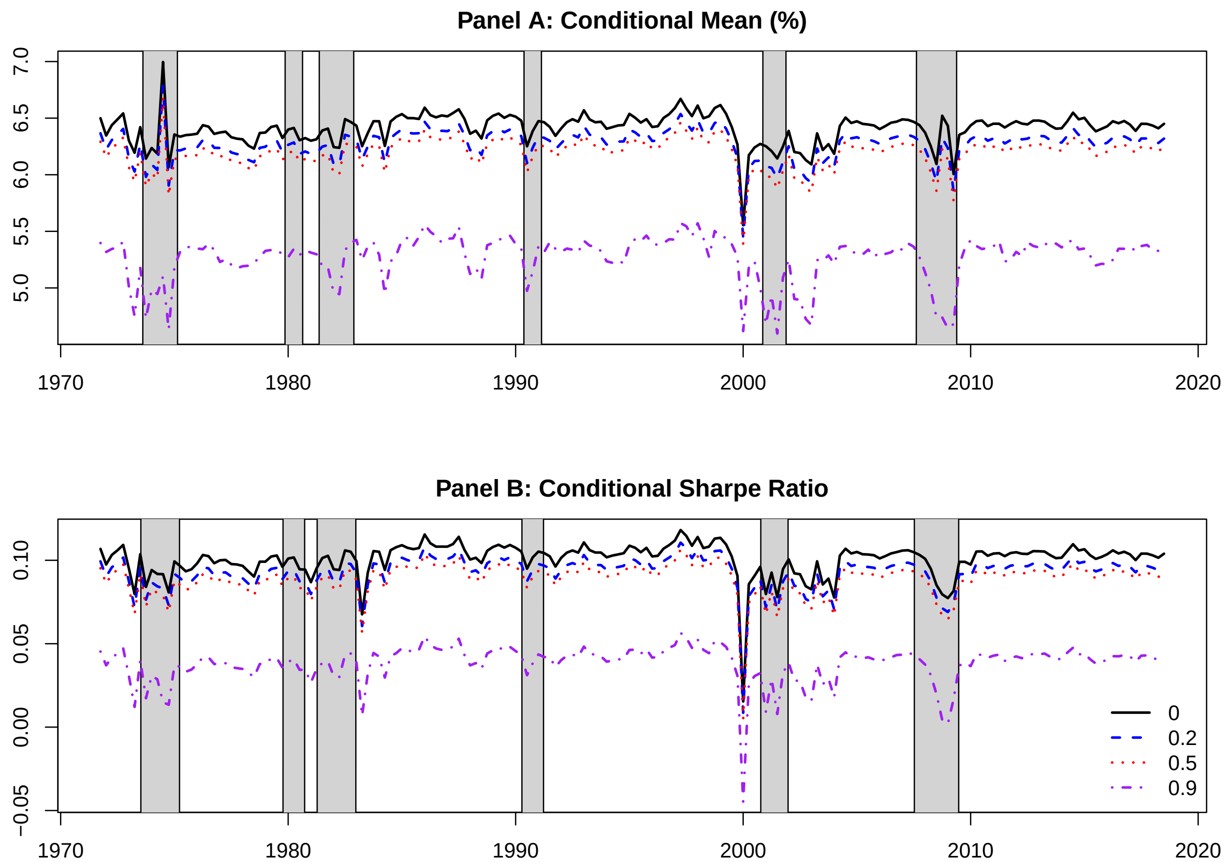
B Appendix: Additional Robustness Results

Figure 10 – Beliefs About the Stock Market Across Different Investor Types



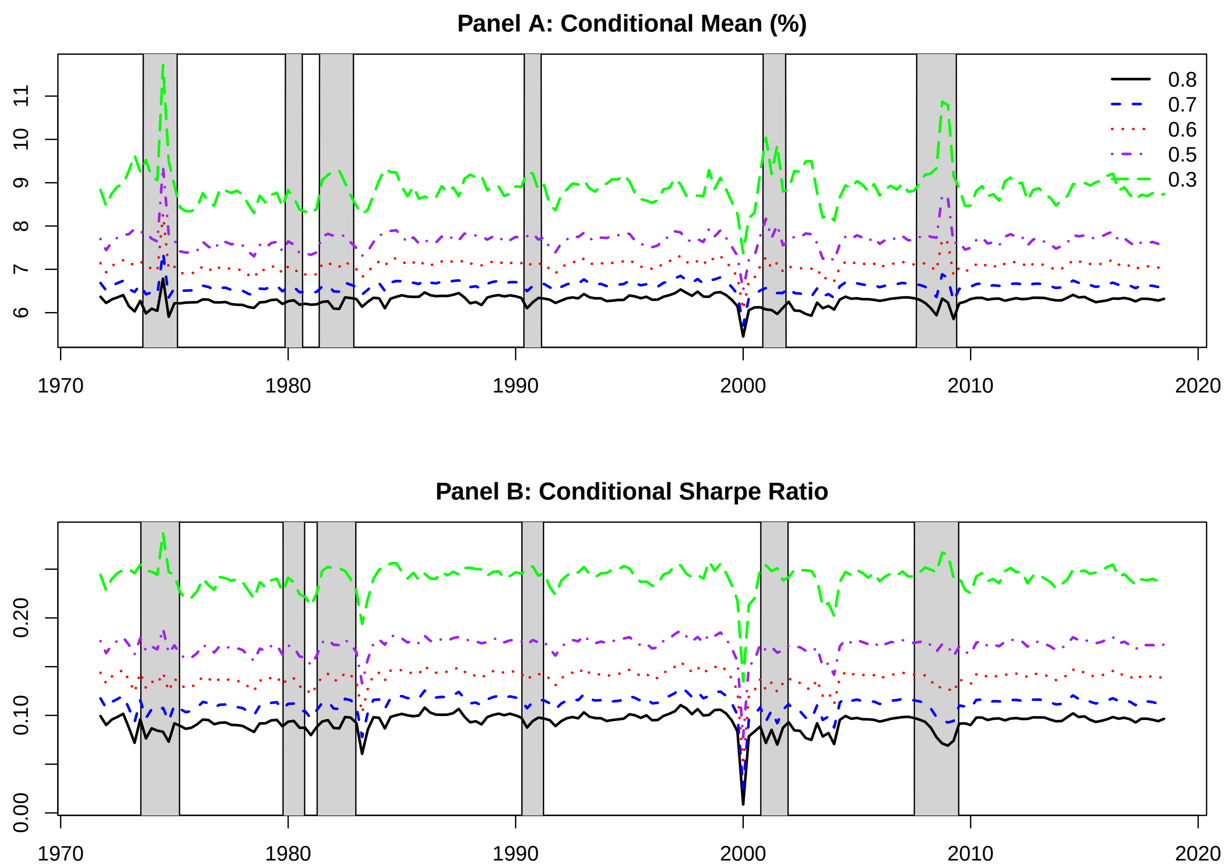
Notes: The figure plots the time series of the conditional mean (Panel A) and Sharpe ratio (Panel B) of the market return, as perceived by four different investor types – those whose portfolio holdings correspond to the small growth portfolio (black solid line), the small value portfolio (blue dashed line), the big growth portfolio (red dotted line), and the big value portfolio (purple dashed-dotted line), i.e. the corner portfolios from the 25 Fama-French portfolios. Shaded areas denote NBER designated recession periods. The beliefs of each investor type are recovered using the SEL approach. For each investor type, the pricing kernel is exponentially affine in their chosen portfolio. The test assets consist of the excess returns on the large cap and small growth portfolios. The conditioning set consists of the averages of the last four quarter's large cap and small-growth portfolio returns. The sample is quarterly covering the period 1972:Q1-2018:Q4.

Figure 11 – Beliefs About the Stock Market Across Investors with Optimal Portfolio $R_W = xR_f + (1-x)R_{SG}$, for Different Values of x



Notes: The figure plots the time series of the conditional mean (Panel A) and Sharpe ratio (Panel B) of the market return, as perceived by four different investor types – those whose portfolio holdings correspond to the small growth portfolio (black solid line), 80% in the small growth portfolio and 20% in TBills (blue dashed line), 50% in the small growth portfolio and 50% in TBills (red dotted line), and 10% in the small growth portfolio and 90% in TBills (purple dashed-dotted line). Shaded areas denote NBER designated recession periods. The beliefs of each investor type are recovered using the SEL approach. For each investor type, the pricing kernel is exponentially affine in their chosen portfolio. The test assets consist of the excess returns on the large cap and small growth portfolios. The conditioning set consists of the averages of the last four quarter's large cap and small-growth portfolio returns. The sample is quarterly covering the period 1972:Q1-2018:Q4.

Figure 12 – Beliefs About the Stock Market Across Investors with Optimal Portfolio $R_W = 0.2R_f + xR_{SG} + (1 - 0.2 - x)R_m$, for different Values of x



Notes: The figure plots the time series of the conditional mean (Panel A) and Sharpe ratio (Panel B) of the market return, as perceived by investor types who allocate 20% to TBills and the remainder between large cap and small growth equities. Specifically, we report results for five investor types – those whose portfolio holdings correspond to 80% in the small growth portfolio and 0% in the large cap portfolio (black solid line), 70% in the small growth portfolio and 10% in the large cap portfolio (blue dashed line), 60% in the small growth portfolio and 20% in the large cap portfolio (red dotted line), 50% in the small growth portfolio and 30% in the large cap portfolio (purple dashed-dotted line), and 30% in the small growth portfolio and 50% in the large cap portfolio (green dashed line). Shaded areas denote NBER designated recession periods. The beliefs of each investor type are recovered using the SEL approach. For each investor type, the pricing kernel is exponentially affine in their chosen portfolio. The test assets consist of the excess returns on the large cap and small growth portfolios. The conditioning set consists of the averages of the last four quarter's large cap and small-growth portfolio returns. The sample is quarterly covering the period 1972:Q1-2018:Q4.